(Transcribed from tape recording by FO 9-22-52) 3rd Lecture
Lecture given Sept. 21, 1952, by FFW

Last Sunday we covered ~~a~~ preliminary ground in which we became, presumptively, more familiar with the meaning of the word "finite", and we, perhaps, all had some experience of an enlargement of our previously existent ideas as to how big finite can be. As a matter of fact we did not deal with any conceptions that were really difficult at all. The difficulty was in the domain of trying to expand the perceptual imagination, to grasp notions which conceptually are rather simple. One lesson that should have come out of that experience is this, that the perceptual power is very definitely restricted. In what we shall do tonight we ~~will~~ shall have to drop the perceptual power and operate with other cognitive powers. I shall outline three cognitive facets or powers.

First of all, "perception," which we shall understand as the cognitive aspect of sensuous experience. The impressions we get from the world ~~get~~ become organized, more or less automatically, into what we call percepts, which are characterized by these qualities, that they are concrete and particular, ~~but~~ and they are also definitely finite in their limitations. Last Sunday we sought to expand perceptual imagination so as to grasp something of the meaning of the googalplex, or $10^{10^{100}}$ ~~that is 10 to the 10 to the 100~~, a pretty big number. The second cognitive power we ~~will~~ shall call "conception." It is a cognitive power ~~that~~ which is non-sensuous in its purity, however much it is true that in common usage it may be more or less confusedly blended with perception. In its purity and in its most efficient operation it achieves a high degree of freedom from the restrictions of the perceptual consciousness. It is characterized by generality, impersonality,

Note to transcriber
This can be typed in by turning the typewriter cylinder down part of a space for each power index Thus ($10^{10^{100}}$)

and definitiveness. While these features are present in variable degrees as among different concepts, yet in their highest development we get an extreme generality and an extreme definitiveness and it is on that level that mathematics exists. The third form of cognition is one that is practically without recognition in the vast bulk of western philosophy and psychology, but not totally without recognition. There are at least references among the German idealists that point to it. By "introception" I mean a cognitive power which transcends the subject-object relationship, but, like "perception," its content, if ~~you~~ we may use that term, is concrete, ~~but~~ while, like unto conception, its content is completely universal, and not particular. Its key word is Light. You might call it cognition as pure Light. In its purity it operates only in the domain of the Infinite. It can be Realized, and when Realized in its purity, the sensuous or perceptual world drops away or vanishes, and likewise the conceptual world drops away and vanishes. There are possibilities of an interblending between these three cognitive components. In our work last Sunday we dealt with an interblending between perception and conception; in other words, we dealt with a domain that is familiar, more or less, to everyone. Tonight, as far as may be, we ~~will~~ shall attempt to drop the perceptual component and its concrete particularity to journey on into the domains in which we propose to enter, and we'll see if we cannot in some measure effect a fusion of the introceptual with the conceptual.

I may say this about the vast majority of mathematicians: ~~that~~ they operate on the level of the conceptual, freed from the perceptual, but without the Light of the introceptual.

¶ When you have the fusion of the introceptual and the conceptual, you have a different domain from that which is familiar to most mathematicians. You have spontaneous luminousness combined with the principle of organization. Now, we have before us a far more difficult task than that of trying to comprehend the googalplex. Let us consider the totality of all natural numbers; these consist of ~~simply~~ the positive integers, the one, two, three, four, and so on beyond all limits. One number and only one in that series is the googal, and another one is the googalplex. Consider this whole series as one entity, that means consider all possible integers whatsoever, — and remember there is no such thing as a last integer, — and embrace that totality as one entity. Now, you cannot embrace it in the sense of putting a circle around it. You could in principle put a circle around the googalplex. The embracing has to be done in another way. Let us say, symbolize it ~~as~~ by the arms held out this way with an open space between the hands and not making a closed circle; the arms thus defining a zone in one sense; the open space indicating a limitlessness. But the task ~~brought to bear on~~ placed before the conceptual imagination now is to grasp that totality as just one entity. We'll have to go further than that. We are indebted to two German mathematicians of the last century for the definite defining and characterization of the infinite. These two are Dedekind and George Cantor. It is noteworthy in the work of Dedekind, in his essay on "The Nature and Meaning of Number", that you hardly ever see in that essay our ordinary numbers at all. It is an essay about sets and classes, about the primary ideas in the mind, and theorem after theorem developing from that simple material, derives the most fundamental properties of number. Some of these we spoke of last Sunday.

Number grows out of the establishing of a one-to-one correlation between two classes. Last Sunday We took ~~it~~ this process back to the stage of the infant and the primitive; we saw how correlation probably first was made with the fingers of the hand and various objects, later with pebbles and various classes of objects like sheep, and so on. That was before notions of number as we have them were born. That is fundamental counting That is fundamental number. The basic notion upon which we build is that we can call two classes similar, or, in ordinary language, equal, when we can set up a one-to-one correlation between the two classes so that there are none left over in either class, Thus, if there were 5 coins and 5 pebbles, we could set up, even if we didn't know the word five or the number five - that notion having not yet ~~hadn't~~ been born~~,~~ we could draw a line between a pebble and a coin and a pebble and a coin etc., and exhaust the two classes at the same moment. When that happens we say they have the same cardinality ~~,~~ the cardinal number~~, which~~ being ~~is~~ the quantity number rather than the order number. We had Better get used to the word cardinality because we are dealing with notions that are very fundamental. And just as an ~~introjection~~ intercallation at this point, I may next Sunday or sometime later deal with some preliminary efforts along the line of what we might call a construction of an Holoistic mathematic - just some preliminary ideas. To achieve any understanding of even the initial idea, you have to grasp the conceptions with which we are dealing tonight. The reasons for that will later appear. But now we are going to note the property that is peculiar of our class of numbers. I put down: 1,2,3,4,5 ~~and~~ (a dotted line afterwards which means it goes on forever) And I'm going to put another line below, which will be the doubling of each of the first numbers.

Note to typist: Write with several dashes and no other writing on the lines.

1	2	3	4	5	—	—	—	—	—
2	4	6	8	10	—	—	—	—	—

Separate lines

1, 2, 3, 4, 5 ——— N

2, 4, 6, 8, 10 ——— 2N

Now here is a very important point. We can set up a one-to-one correlation between these two classes or sets - thus: ~~at this line here~~ - That's counting. If there are just as many in one set as there are in the other, they have ~~are~~ the same cardinality. Now, is~~n't~~ it not evident to you, that no matter how far we go in the first set we will always have a number in the second set corresponding to each number in ~~that~~ first set ~~in the second set~~, and corresponding to any number "n" in the first set ~~over here~~, there will be a "2n" in the second set ~~here~~. There will always be a 2n corresponding to the n; therefore there are just as many numbers, just as many elements, in the second set as there are in the first set. But there is another important fact - every element in the second set is to be found in the first set. Two is found over here, four over here, six over ~~in~~ there, and so on, in the first set, Yet there are elements in the first set that are not found in the second. One is not found in the second set, three is not, in fact, every odd number is not found in the second set. There are just as many in one set as the other; the totality of elements in the second set is the same as the totality of elements in the first set. They have the same cardinalty. The second set is a sub-set of this, because all of it is found in that, but not all of the first set ~~that~~ is found in the second set ~~this~~. Now that quality, that property, is the definition of an infinite class. An infinite class is a class which has one or more parts~~,~~ - proper parts~~,~~ - which have the same cardinality, that's the same number~~,~~ totality, as the whole. ~~A proper part which has the same cardinality as that of the whole.~~ That is never true of any finite collection, or finite class. You take a proper part of a googal, for instance, you take ~~take 100~~, a sub-set of 100, out of that googal and the googal will be reduced in its cardinality by that 100. You can't set up a one-to-one correlation.

It does not have a proper part, which has as many elements in it as the whole. Only infinite classes or sets have this property.

Now, there are some very wonderful things you can do with our integers. Would you believe that you could count with the integers all the rational fractions? Just consider the rational fractions between zero and one. It is obvious, isn't it, at once that there is an infinity of them in ~~there?~~ that domain? One over a googalplex would be one of the fractions in there, one-half, one-third, all the fractions with one in the numerator and any number in the denominator, and several with a larger number than one in the numerator, and that between one and two you find a similar infinity, and so on between ~~every~~ all contiguous integers whatsoever ~~out from~~ of the whole series. You would have an infinity of fractions between every one. Is that clear? ~~Any one to whom it is not clear?~~

Now what we propose to do is to count the sum-total of all fractions in the whole number system ~~that goes~~ which extends out to infinity. What do we have to do to do that? We have to order ~~them,~~ the elements in a definitive way, such that, we~~'ll~~ will be sure of ~~picking up~~ counting every fraction whatsoever. Think about it. How would you go about that? How would you start a system that would enable you to know certainly that in that system you had all of your numbers — rational numbers, fractions and integers, so ordered that you had them all and knew that you had them all. Now you could~~n't~~ not start from zero and ~~say you~~ then take the next fraction. It wouldn't be one-half, it wouldn't be one over a googal, it wouldn't be one over a googalplex. There~~'s~~ is an infinity of ~~them~~ fractions between a googalplex and zero. Now we want to try to order ~~them~~ the rational numbers so that we can start counting. You can~~'t~~ not count until you can order. It so happens that.

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this is worked out in a very clever ~~way~~ and ~~a~~ rather simple way. Let us write the numbers in this fashion: . . . . (Introduce attached page

You write every number as a fraction. Start in here, one over one. Now the numerator will always be ones on this line. The denominator will correspond to the number there - one over three, one over four, on to infinity. Down here we will have two over one, two over two, two over three, two over four, and we'll have three over one, three over two, three over three, three over four. Down here we have four over one, four over two, four over three, four over four, and so on. Well, you follow that system out - clear to infinity in every direction - and you will get every fraction that there is in an orderly arrangement, including every whole number. One over one reduces to a whole number, two over two reduces to one, three over one reduces to three. You will have every integer, positive integer, you will have every fraction whatsoever somewhere in that system, and they are nicely ordered. This will show right here. All of these continue on to infinity - this way and that way. Suppose we start and instead of writing it that way we write it one and a comma and one; that stands for that fraction; it is just a different way of writing it. From here we go over here. We write (1,2) and then we come down here on a diagnal and pick that up, we get (2,1). Then we go back over here and we get (1,3), we get (2,2), and (3,1) and so on, and then we can set up a one-to-one correlation. One - tied in here. This is the second one. This is the counting process. This series, you can write this very simply now, if you notice certain rules. The sum of the two keeps ascending. There'll be some with the same sum, that is, in here one and two are three and two and one are three. You write them in the order of the ascending numerators, hence you have perfect order. You see, this numerator is low, this

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one is larger; then you take the ascending numerators here, the first number being the numerator in each case, the second number being the denominator in each fraction. Now you see that here you will actually pick up every rational fraction there is, that you have given to it a definite order, and that now you can count it. And no matter how far you go out here, you always have a whole number, an integer that will correspond to your rational fraction; hence there are enough integers to count not only all the integers, but all the integers and all the rational fractions in addition. This is mathematics of the infinite now; not the mathematics of finite ~~quantities.~~ manifolds or classes. It~~'s~~ is a different ~~dimension~~ domain developed by with a mental process. But it just so happens that this correlates with and gives a rational pattern to, many reports from mystical experience --experiences that appear to the ordinary consciousness as quite irrational ~~when~~ as they are ordinarily formulated. When you use this kind of logic, they fall into a comprehensible and rational form. ~~That's where the importance of this comes in.~~ This is what makes this discussion of the infinite important. In fact you do not have to say "thinking has to stop when ~~you~~ one gets over into at least some dimensions of the Transcendent". We are dealing with an instrument that enables us to carry a kind of thinking over into the Beyond ~~— that's where the importance of following this kind of reasoning comes in~~. What we are using here in our one-to-one correlation is precisely what primitive man did when he counted with his fingers. And if you are justified in saying that if you get a correspondence with these five fingers ~~among~~ and certain objects, then ~~you are justified in saying~~ they have the same cardinality, then, also, you are justified in saying the totality of all rational numbers has the same cardinality as that of all integers. ~~Using the same process, exactly the same process, just as rigidly, we can say of this series here that it has the same cardinality as that. The series or collection consisting of all whole numbers plus rational fractions can be counted by all of the whole numbers.~~

You've got to forget all the rules that held in your ordinary grammar-school arithmetic. This is another domain. Now this infinite, an infinite like this that can be counted, is called a denumerable infinite. The idea is that if you could count for an infinite time you could count them all. Later we will have to consider the infinites that cannot be counted.

Now we have another thing. This is a simple one. This is not severe yet. The next one we will attempt - I will ~~just~~ merely ~~point out~~ note the fact that a further proof was made. that demonstrated that, not only the whole of the rational numbers, but also the whole of the algebraic numbers can be counted. Algebraic numbers include all rational numbers, plus a large number of irrationals like the square root of two and imaginaries like the square root of minus one, or complex numbers like ~~a into the square root of minus one plus b~~ $a + b\sqrt{-1}$. They are numbers, the technical definition of which, you probably would ~~n't~~ not understand, and would ~~n't~~ not be expected to understand. But they are the numbers that can be the solution of algebraic equations of any degree having integral coefficients. The class of numbers is so large that we ordinarily represent them by a plane or two-dimensional ~~plane~~ space. We put our ordinary numbers, which we call commonly the real numbers, just a name, and so on (demonstrating) - now we have the minus numbers in the opposite direction, and we have on this vertical line the imaginary numbers. We have minus the square root of minus one, minus two times the square root of minus one, minus three times the square root of minus one, and so on indefinitely in each case, and out here we would have points in space. This number would consist of three - one, underneath like that, plus two into the square root of minus one - that is called a complex number - and this space

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This is done by the method illustrated in figure I:

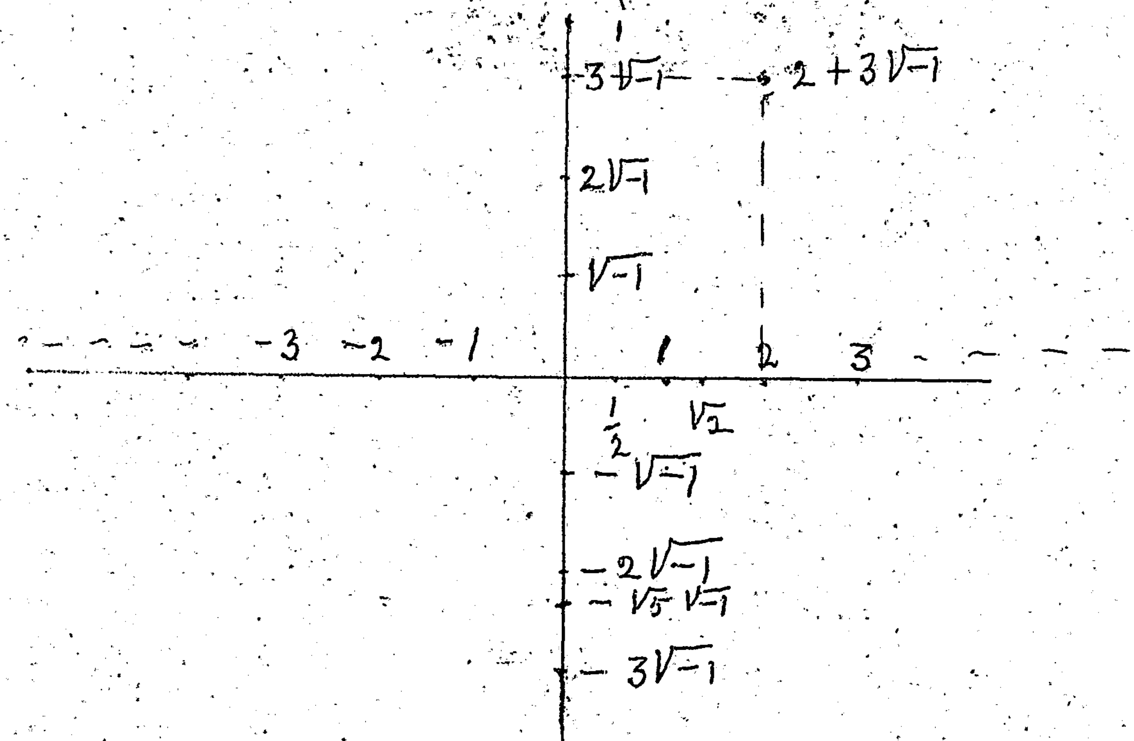

Fig. I

Two lines are drawn at right-angles to each other, one horizontal, the other vertiwal, as in the rectilinear co-ordinate system. An arbitrary distance along each line is given the value of unity and the integers associated with multiples of this unit, positive integers to the right and negative integers to the left, on the horizontal line. Fractions, such as $\frac{1}{2}$, and the ordinary irrationals, such as $\sqrt{2}$, are associated with their appropriate points between the integers. The immaginary numbers -- involving multiples of $\sqrt{-1}$ -- are similiarly associated with points on the vertical line, with fractional and irrational multipliers appearing in their appropriate positions. The numbers appearing on the horizontal line are known as "real" numbers, on the vertical line as"immaginary" numbers. Numbers which are formed as an algebraic sum of a real and an immaginary number are called "complex" numbers by points of the plane, as indicated in figure I.

Numbers of the foregoing type can, in general, be solutions of

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of algebraic equations. It is clear that we have now added several infinite classes to the class of the positive rational numbers, i.e., the negative rational number, the ordinary irrationals, the immaginaries and the complex numbers. Yet Cantor proved, by a method which we shall not review here, that the sum-total of all these numbers, which can be solutions of algebraic equations, can be ordered in such a way that a one-to-one correlation can be set up between them and the sum-total of positive integers. Hence the totality of all these numbers is denumerable.

would be filled all full with every possible number, with every possible fraction whatsoever. We take that up as a convenient way of representing them. The   made by Cantor, that the sum total of all these numbers that can be solutions of algebraic equations can be counted by the rational integers. We won't try to do anything more than just state that fact.

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But we come now to the next step: A proof, although there is at this point, some difference of opinion, that ~~you~~ one cannot count the total of all real numbers. Real numbers consist of those that are not imaginary like our integers, like our fraction ~~a over b~~ a/b, and all of the simple irrationals, such as the ~~cube root of 7~~ $\sqrt[3]{7}$, and so on. ~~all of these that I put on the board here can be solutions of algebraic equations with integral coefficients~~, But the real numbers include numbers like ~~Pi~~ π, the ratio of the diameter to the circumference of a circle, and ~~E~~ e, which may be written $\lim_{n=\infty}\left(1+\frac{1}{n}\right)^n$ ~~the numbers written this way (demonstrating) one of the simple ways - the limit when n equals infinity of one over one to the nth power~~. These ~~Two~~ two numbers of enormous importance, ~~Pi~~ π you can appreciate. ~~E~~ e (is) the base of our natural system of logarithms, but, more important than that in one respect, ~~I~~ it is found wherever you study the phenomena of life. Get the statistical data connected with anything that is living, draw your curves that correspond to your statistical data giving your life cycle, the this curve, when reduced to a formula or to an expression, always involves the number ~~E~~ e. There is some mystery in that. But ~~E is~~ e may be called the number of life. Now ~~E~~ e and ~~Pi~~ π are transcendental numbers. That means technically that they cannot be solutions of algebraic equations having integral coefficients. At the time of Cantor these were the only transcendentals definitely known. ~~They are an awfully hard number to find~~. But his proof was that transcendentals ~~they~~ are so much more numerous when contrasted with algebraic numbers that ~~when added to the other numbers~~ they cannot be counted.

Now here is a small example ~~little bit~~ of proof that ~~begins to take you into the domain of higher mathematics~~ employs, the reasoning that is sometimes used in higher mathematics. We are presented with ~~Facing~~ the problem of ordering all the real numbers. That does ~~n't~~ not mean merely the rational numbers, integers and fractions, which it is easy to order, but now we ~~you~~ are going to try to order all the real numbers. That means we ~~you~~ would have to include ~~get in~~ every irrational and every transcendental there is. ~~would be~~. It is an impossible order. There is no way of doing it - at least no way that a human mind can envisage. Cantor suspected that the number of real numbers was so great that one ~~you~~ could not count them, even with an infinity of integers. Now let us consider the region from zero to one. If we could prove that one ~~you~~ could not count all of the real numbers between zero and one then obviously it would be impossible to ~~you couldn't~~ count all the real numbers from zero to infinity. So all we have to consider, if we are going to prove that they can~~'t~~ not be counted, is the region from zero to one. Let us take every number ~~fraction~~ ~~these would all be fractions~~ and write it ~~them~~ as non-terminating decimal~~s~~. Thus, ~~for instance,~~ while some are naturally non-terminating, most of them would be, and if we ~~you~~ had a decimal, such as .4 ~~four~~ which is complete, then we write it in the non-terminating form ~~you would write it this way~~ .399999 ~~forever and your 9s go on to infinity and when you get to infinity this number is just as big as the .4~~. So you can write every one of these ~~those~~ terminating decimals or fractions in a non-terminating form.

We are going to write all the numbers between zero and one in a non-terminating form; but since ~~yet~~ we can~~'t~~ not find an order, so we simply assume that an order exists. We write our first number - let us use letters to represent the digits in our fractions we have a-sub-one, a-sub-two, a-sub-three, and so on to infinity.

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Then we compose a Table in which we represent the infinity of non-terminating decimals by employing letters with subscripts to stand for the digits in each decimal, as is given in Table III.

$$
\begin{array}{l}
1 \leftrightarrow 0.a_1a_2a_3a_4a_5 \ldots\ldots\ldots\ldots \\
2 \leftrightarrow 0.b_1b_2b_3b_4b_5 \ldots\ldots\ldots\ldots \\
3 \leftrightarrow 0.c_1c_2c_3c_4c_5 \ldots\ldots\ldots\ldots \\
4 \leftrightarrow 0.d_1d_2d_3d_4d_5 \ldots\ldots\ldots\ldots \\
5 \leftrightarrow 0.e_1e_2e_3e_4e_5 \ldots\ldots\ldots\ldots \\
\vdots \qquad \vdots \; \vdots \; \vdots \; \vdots \; \vdots \; \vdots
\end{array}
$$

Table III

We set up this Table as indicated and establish a one-to-one correlation between the positive integers and the series of non-terminating decimals. If our Table embraces all of the real numbers between zero and unity then we would have proven that they are denumerable. But examination reveals that no matter how completely we develop the set there always remains an infinity of numbers which have not been included. This is evident from the following consideration: If we write a non-terminating decimal which differs from the first decimal by having a different digit or one other than a_1, in the first place, and other than b_2 in the second place, and other than c_3 in the third place, and so on, then this number will differ from every number in the Table in at least one place. This process can be repreated by diagonals beginning with a_2, a_3, a_4 and so on, so that obviously there would be an infinity of numbers not included in the Table, however complete we attempt to make it.

Now our next one - we are just saying arbitrarily that an ordered way has been found - maybe only that God himself could find that order - but we just say that it exists, it exists somehow and we let these letters stand for it. This is where the reasoning gets subtle. We have c-sub-one, c-sub-two, c-sub-three - (demonstrating) - now this goes on down here to infinity too - infinity in both directions. Assuming that we have been able to order them and count them, we examine it to see if there is any difficulty, and at once a difficulty arises, because we can find that there is an infinity of numbers no matter how we arrange this that will not be included. For instance, consider the numbers that you get by taking that diagnoal down here - write another number where you change this digit in the series and change that digit, change that digit and so on clear through, you will clearly have a number that will be different at least in one place from any number that you may have in this series. Can you see that? That means then that you have not got all of the numbers in. As a matter of fact, since you could do that on any of these diagnnals there would be an infinity of numbers that you would not have included no matter how far or completely you wrote this out. In other words, our assumption that ~~you~~ we could order ~~them~~ the numbers and count them has proven false. The other conclusion is that they are not denumerable, that they are so numerous that the infinity of digits that could count all our fractions, all our algebraic numbers, ~~and could count all those numbers where you multiplied them by infinity~~, still ~~could count them,~~ could not count the sum total of all real numbers. Here is where the logic ~~gets~~ becomes subtle.

The logical principle that~~'s~~ is employed here is this: first we say that the totality of all real numbers is countable or it is not countable. If we ~~you~~ find that when we ~~you~~ assume that it is countable we ~~you~~ run into a contradiction, then the conclusion must be that it is non-countable. That's the dichotomy. The question ~~Questioning~~ as to whether this reasoning is sound or not depends upon whether the dichotomy is valid. Thus, for instance, if we ~~you~~ were to say that an equation is either reducible or not reducible, we ~~you~~ would have two possibilities ~~classes~~. It belongs one class ~~way~~ or the other. Is this principle valid? ~~Does that hold?~~ Is there some middle ground which belongs to the zone of that which is not reducible and not not-reducible? Some criticism of the reasoning here has been brought from that angle, but if we accept the soundness of the reductio ad absurdum then it follows that the sum total of all real numbers is more than a denumerable infinite. Now here is the interesting fact. In Cantor's time two transfinite numbers were known. Since then several classes of an infinite number of transfinite numbers have been discovered. They are ~~immeasurably~~ infinitely more numerous than all the other numbers put together, and yet they are hard to discover, and only two of them are well known to everybody, namely π ~~Pi~~ and e ~~E~~.

Let us suppose we took all numbers other than the transcendental; all the algebraic numbers: integers, rational fractions, ~~and~~ the ordinary irrationals, ~~and~~ the imaginaries and complex numbers, and we placed them out in space as I showed before in Fig. I. Then we find this true, that between any two of those numbers that would correspond to specific points ~~like the~~ points I ~~have listed~~ here which we will ~~say are a into square~~ root of minus one plus B and ~~C into~~ the square root ~~of minus~~

~~one plus D~~ - between any two points ~~you~~ we can always find another number. Do you see that from that statement it follows that ~~you~~ we can always find an infinity of other numbers? Here is a check of our logical sense. If between A and B, or 1 and 2, ~~I~~ we can find another number, if all this is true that between any two numbers ~~I~~ we can find another number, then it must follow that between those two numbers ~~I~~ we can find an infinity of other numbers. Of course, quite obviously, there exists c between ~~your~~ our ~~C~~ a and ~~your~~ our ~~B~~ b another number, ~~but~~ which we call ~~that~~ ~~C~~ c, but our rule ~~would~~ say~~s~~ that between ~~C~~ b and ~~B~~ c ~~you would~~ we can find another number, and so on ad infinitum. That is another feature of mathematical thinking that is very fundamental. It's part of the step from any-ness to every-ness.

If ~~you~~ we can say something about any member of a class, or set, or group, or collection; (by "any" we mean one picked at random), then whatever we can say of 'any' we can say of 'every'. You see we are picking out 'any' on the basis of its general property, and are not concerned about particularities that may attach to special entities. Now it would seem, would it not, to you that after we got down all of the algebraic numbers, all of these numbers we have been talking about except the transcendental~~s~~, that that plane would be pretty solidly filled, ~~wouldn't it~~? ~~R~~remember~~ing~~ that ~~you~~ we can always get an infinity of numbers between any two points. ~~Y~~yet as a matter of fact that plane would be like a sky with the numbers corresponding to points like stars with vast blank spaces in between. ~~Your~~ Our plane is not ~~tightly~~ densely filled. Remember ~~your~~ our points have no area at all. They're absolutely sharp, area-less. They have~~n't~~ not packed that plane, but actually that plane has infinitely greater spaces in it than the space that would correspond to the numbers. In other words, without the transcendentals ~~you~~ we do~~n't~~ not have a true continuum.

Note to transcriber, Write in the Greek and Hebrew letters with the name in parenthesis

The only way ~~you~~ we can fill that space is by bringing all of the transcendentals into it. I think you can begin to see the enormous vastness that belongs to the transcendental numbers as compared to all of the other numbers. So, one simple notion of infinity is not enough to take care of our total problem of determining the cardinality of all possible classes. This leads us to what you might call a heirarchy of infinities.

The first infinity ~~that~~ which corresponds to the total of all integers, which was sufficient to count integers and fractions and in addition sufficient to count all algebraic numbers, has been written variously as $\aleph_0$ (aleph null), or aleph sub-zero) and sometimes ~~has been written~~ as ω (omega) sub-zero. ($\aleph$ ~~One~~ is the first letter of the Hebrew alphabet and ~~the other~~ ω is ~~about~~ the last letter of the Greek.) And this is known as the denumerable infinite, corresponding to the cardinality of all integers. The cardinality of all real numbers is more than infinitely greater than that. We have a very interesting multiplication table or certain laws that attach to these numbers. We take $\aleph_0$ ~~aleph~~, add ~~one~~ 1 to it, and the answer is just $\aleph_0$ ~~aleph sub-zero~~. We Add a googalplex ($10^{10^{100}}$) to it, and it swallows ~~it~~ that just as easily. The answer is still $\aleph_0$ ~~aleph sub-zero. Just swallows it as easily as it does the one~~. Or again, if we subtract a googalplex from it, we have $\aleph_0 - 10^{10^{100}}$ ~~aleph sub-zero minus a googalplex, which is ten to the ten to the 100,~~ and that just equals $\aleph_0$ ~~aleph sub-zero.~~ You can~~'t~~ not disturb ~~its~~ the calm of $\aleph_0$ in that way. You see, a whole universe like this we live in could drop out and ~~it~~ $\aleph_0$ would go on just as placidly as you please. Nothing happened. Or, now let us try something else and see what multiplying ~~would~~ will do. We~~'ll~~ will multiply ~~it~~ $\aleph_0$ by a googalplex, ($10^{10^{100}}$) ~~10 to the 10 to the 100 times-- I can't justify these statements, you'll just have to take them~~ -

still $\aleph_0 \times 10^{10^{100}} = \aleph_0$

~~I think that's enough anyhow. - what happened?- swallowed it up just easily~~ $\aleph_0$ has~~n't~~ Not changed it at all. The multiplication table is very easily learned when it's like that!

We can go still further ~~Another thing, you take~~ and multiply ~~it~~ $\aleph_0$ by itself; that~~'s~~ is equal to $\aleph_0^2$ ~~aleph squared~~. It just equals ~~aleph sub-zero~~ $\aleph_0$. We haven't disturbed it yet. It takes something more powerful to disturb it than that. ~~That~~ This means that none of these processes have taken us out of the domain of the denumerable infinite. This is what ~~you~~ we have to do to ~~have~~ produce any effect on ~~that number~~ $\aleph_0$ - we raise $\aleph_0$ to the $\aleph_0$ power ($\aleph_0^{\aleph_0}$) ~~you take aleph now raised to the aleph null power~~ and at last that does something, ~~I you~~ We get ~~aleph sub-one~~ $\aleph_1$, the second transfinite number. Now, you may ask, ~~say~~ does this correspond to anything? It corresponds to the cardinality of the totality of all real numbers including the transcendentals, and the cardinality of the continuum; that is, the mass of numbers it would take to make all of this space solid.

¶ ~~And~~ The same effect of multiplication and addition applies to ~~aleph sub-one~~ $\aleph_1$. As a matter of fact, $\aleph_1$ ~~aleph sub-one~~ raised to the ~~aleph null~~ $\aleph_0$ power remains unchanged ~~just swallows that up and~~ it remains aleph sub-one. It~~'s unaffected by it~~. The only ~~thing~~ operation that affects $\aleph_1$ ~~it~~ is raising it to the ~~aleph sub-one~~ $\aleph_1$ power in which case it achieves a higher cardinality and becomes ~~aleph sub-two~~ $\aleph_2$.

No ¶ Now, there is some evidence that ~~this~~ $\aleph_2$ corresponds to a class of entities with which we actually deal. The statement is that it corresponds to the number of single valued functions, but you ~~won't~~ will Not understand that, ~~and those so far as I know are the only ones that have usage~~. ¶ Let us assume the process carried to the limit, and we get ~~this~~ $\aleph_\infty$ ~~(aleph sub-infinity)~~ - the symbol of the whole. The Holoistic. This is the most comprehensive conception evolved in the mind of man. And since the mind of man is a part of the Whole it could not evolve something greater than the Whole, hence the most adequate symbol of the vastitude of the whole is ~~aleph sub-infinity~~ $\aleph_\infty$.

~~Your~~ Our googalplex by now is a microscopic pellet. In the sea of the Illimitable, the whole galactic universe, nay, a denumerable infinity of galactic universes of the same size, would dissolve into ~~a~~ sub-microscopic insignificance. It really makes no difference whether you call the universe an Illusion, as Shankara did and as the Buddhists do, or whether you call it Real, as Sri Aurobindo does. In any case, in the presence of the multiple infinitude of the Whole, ~~they~~ it ~~are~~ is absorbed as an insignificant irrelevancy. Hence, whether it is real or an illusion is not a point of vital importance.

When a mathematician speaks of the infinite, he does not mean merely a big number. He means things like this series of which we have spoken. But he means in differentiating between infinities of different orders that they still have a character, that it is not a blank of largeness in which there is no element of determinateness at all, but rather that they have a character so that there is something distinguishable - a heirarchy of infinities. Now the question would arise, how could a finite creature ever know, ever realize, the Infinite. And the answer is, a finite creature never could, for the finite creature would be limited to a progression of finite steps, and in a finite time could never realize the Infinite. But if the reality of man, nay the reality of all creatures, of all entities, is that they are part and parcel of the Infinite, not merely cut off apparent finite fractions, but co-extensive with the Infinite, then the Infinite is knowable in the sense of Realization by the simple removal of an obscuration. ~~- the seeming of finitude being placed by some instrument of obscuration.~~ I considered it very significant when Dedekind gave his existence-theorem concerning the reality or existence of infinite manifolds.

~~He~~ he said, take the ideas in the human mind. ~~You~~ One can have an idea which we call ~~a-sub-one~~ a_1 and then ~~you~~ We can have an idea of that idea, which we'll call ~~a-prime~~ a_2, and then the ~~a-prime~~ a_2 can be put in the first series as an object of thought and ~~your~~ our ~~a-second~~ a_3 would be the idea of this idea, and that can be placed up ~~there~~ in the first series and the process be continued in that manner. Every idea in the second series would be in the first, but there would be one idea in the first series that is not in the second. Particularly he gives the idea of our own ego as one not included in the second. ~~You have a cardinality, equal cardinality~~ Both series have, the same cardinality because of the 1-to-1 relationship; therefore, the ideas in the human mind are infinite. Now that does~~n't~~ Not mean that they are infinite in the sense of actual concrete thinking of an infinity of ideas. You might say it is infinite by this power of a generat~~t~~ing progression. But the very power to generate the progression and to see it points to its infinitude. I know these ideas have some subtleties in them. They are not too easily grasped. I am quite sure that the lecture of last Sunday probably seems rather simple now, and the googalplex is something you may take in your stride relative to this.

I have been thinking during the last few days of a possibility of formulating the first principle of what we might call a holoistic mathematic, and I might by next Sunday be prepared to give a first talk on this, but I~~'ll~~ will have to assume that you are familiar with the kind of thinking we've been doing tonight. It will prove necessary if we are going to use the basic holoistic conception, to use the mathematics of the trans-finite. This that we have done tonight is preparatory in one sense to ~~that~~ this new conception, ~~as the other part of its meaning was to give~~ Our other purpose was to secure some more adequate understanding of what is meant when we speak of the Whole. This is No simple denumerable infinite but a vast non-denumerable Infinity, compounded an infinity of times. Naturally, we sink as relative beings into a less than microscopic significance compared to That, but he who knows that this ~~vast Thisness,~~ Vastness which is none other than Parabrahm, is That with which in truth he is ~~I am~~ identical, need not identify himself with an insignificant finite appearance but may know, as Shankara said, that he is not only a part of Parabrahm but that he is identical with the Whole of Parabrahm.

Now let us add to that, Sri Aurobindo's insistence ~~of~~ upon individuality. By the use of the conceptions we employed to-night it is quite readily possible to reconcile those two statements of Identity ~~of~~ with Parabrahm or the whole of the Holoistic, and yet retain infinite variety of infinite individuality. That, I think, is enough for tonight.