Infinity and Expanding Relativity

Introductory Essay

Saying ascribed to Buddha:

> "How can there by any understanding or teaching of that which is wordless (i.e., inexpressible)? That can be understood and taught only by Samāropa - an ascribed mark."

The Conception of Buddhist Nirvana by Th. Stcherbatsky with comprehensive Analysis and Introduction by Jaideva Singh, p. 51 (Gordon Press, N. Y. 1973)

> "Śūnyatā * was declared by the Buddha for dispensing with all views or 'isms'. Those who convert Śūnyatā itself into another 'ism' are veinly beyond hope or help"

(Nagarjuna: Madhyamaka Kārikās 13,8) translation in Stcherbatsky, ibid p 42, p. 82)

Buddha to Kāśyapa:

> "If a drug administered to a patient were to remove all his disorders but were to foul the stomach itself by remaining in it, would you call the patient cured? Even so Śūnyatā * is an antidote against dogmatic views, but if a man were to cling to it for ever as a view in itself, he is doomed".

ibid p 43, and main text p. 82.

* Śūnyatā = Expanding Relativity (our translation):

"Expanding" because of its etymological derivation from the root "svi" meaning "to swell, to expand"; the same meaning holds for the root 'brh' or 'brnh' from which Brahman is derived! "Relativity" because it is "the theory that nothing short of the whole is real (read: the same for all observers), the parts being always dependent are ultimately unreal". The Conception of Buddhist Nirvana, p. 36 and index, p. 52)

Elsewhere, saying ascribed to Buddha:

> "Śūnyatā is to be treated like a ladder for mounting up to the roof of prajna**. Once the roof is reached the ladder should be discarded."

ibid p. 43

** Prajna = Transcendental Wisdom

Infinity and Expanding Relativity

"...pure mathematics, in which the discerning student will find veiled the Wisdom Religion, may serve as a means to the Realization."

from the early book "YOGA, Its Problems Its Philosophy Its Technique" by Franklin Merrell-Wolff under the pseudonym Yogagnani (Skelton Publishing Co., Los Angeles, 1930).

Great clarity, great precision, great honesty - such are the characteristics of the individual man Franklin Merrell-Wolff. Since 1936 there is added a transcendental component. The author of this book He is one of our precious fellow beings in whom has occurred a transfinite widening of consciousness. "Recognition" was his name for it in his first major book[1] which was a personal record of transformation in consciousness. He would now describe this as the opening up of the faculty of "Introception". The evidence is overwhelming that these transformations actually occur in individuals. Further - as with the experiences which transformed Gautama to Buddha, or Jesus to Christ - they may affect human history to an extraordinary degree. Both the yearning for and realization of transcendental states of consciousness seem then to be individual and social facts, though it appears up to the present that the number who yearn far exceeds the number who realize. The question then arises for the "usual" human consciousness: how to live with this radical challenge - whether to deny or accept the reality of transformation of consciousness, and whether the denial or acceptance be unconditional or conditional. In view of the diversity of structures in which humans are immersed - religious, political, social, economic,

1) Pathways Through to Space by Franklin Merrell-Wolff (Richard R. Smith, Publisher, New York 1944; reprinting by Julian Press - now Crown - New York 1973; paperback reprinting by Warner).

academic and others - one can hardly speak of a single "usual" human consciousness. Nevertheless if one limits consideration to the generally recognized major psychological faculties - perception and conception - the issue becomes more clearcut. How are the usual perceptions and conceptions, even highly evolved and refined, to accommodate what is purported to be a third organ of cognition - "introception", in the apt nomenclature of Franklin Merrell-Wolff? Certainly the highly symbiotic perceptive and conceptive faculties do not easily give affirmation to this mysterious third faculty which is strictly neither perceived nor conceived. Here we must steer a middle course between wielding Occam's razor and arrant reductionism on the one hand and extending naive credulity on the other. Or, in simpler language, we must maintain some balance between a sweeping "not proved - nothing but" attitude and an "everything claimed is true" attitude. Though we eschew the latter naive credulity we must honestly own that we personally do take the position of "resolute credulity".

It will help remove some critical doubts if we recognize that our perceptions and conceptions constitute the <u>contents</u> of consciousness whereas the presumptive third - introceptive - faculty might relate to the <u>context</u> or "screen" of consciousness on which the contents are projected. It is thus possible in principle that, with the perceptual and conceptual contents reduced to a low noise level, a sensitized awareness of the context or screen could manifest - introception. This would be non-interfering or orthogonal to the contents. In the limit of zero noise, introception would be associated with "pure" consciousness - "Consciousness Without an Object", the title and subject of the author's book[2)] immediately preceding this one. (The present volume is a continuation which may

2) <u>The Philosophy of Consciousness Without an Object</u> (The Julian Press, Inc. - now Crown - New York 1973)

be read independently.) However much such remarks are evocative of the possibility of this fresh and wonderful cognitive faculty which opens the door on the previously unknown, they do not necessarily compel conviction of its existence. They do not strictly remove the conceptual or intellectual obstacles even to conditional acceptance. Only direct experience would give certainty. Such experience would establish the conditional existence of introceptive pure consciousness; we would say further that it exists unconditionally if it could be shown that it is potentially within the range of all consciousness - human, and in whatever other form it may occur within the cosmos. Notwithstanding these reservations, clear powerful and precise analysis, speaking the language of conception itself, can lower and even remove the conceptual obstacles. This might be done individually in the realms of various major categories of thought - philosophical, mathematical and in the domain of the natural sciences. With the obstacles removed by effective analysis and parallels, these conceptual domains become allies in opening the door.

The present book, along with the larger work of which it is a part, is in my opinion a major landmark in the history of philosophy. In it the author accomplishes the indicated task of removing the obstacles to introception in the realm of philosophic thought. His qualifications for this task are impeccable. As a young man, having completed his studies and already teaching philosophy at Harvard he renounced the prospective academic career when he grew convinced that this renunciation would facilitate his movement towards realization. This conviction was vindicated on August 7, 1936 when there began the profound series of transformations of consciousness described in his subsequent writings[1)2)]. In the ensuing years he has subjected these experiences to a thorough philosophical analysis. These remain as possible allies in the effort to open the door to introception the domains of pure mathematics and of the natural sciences.

Of the two, the former stands on its own foundation as an apex of human reasoning and also appears as the conceptual language of the sciences - certainly in the physical sciences and increasingly in the biological and social sciences. Hence the motivation for this essay in which we wish to give convincing analogues, metaphors, or paradigms, in modern mathematics, for introception. Dr. Franklin Merrell-Wolff (again by academic education, as well as a great natural gift for precision) is highly sympathetic to pure mathematics, and confirms ~~into~~ its supportive role for introception. He indicates this in several places. In chapter IV on the New Realism he writes:

> "(c) The third view, which is here called the "gnostic", maintains that mathematical, and therefore logical, knowledge is essentially _a priori_, by which is meant that it exists independent of experience. However true it may be that this knowledge does not arise in the relative consciousness, in point of time, before experience, yet it is not derived from experience, however much it may employ a language which is derived from experience. It is thus in its essential nature akin to mystical cognition - and hence gnostic in character - rather than similar to empiric knowledge."

In Chapter VII on Idealism he writes:

> "But when mathematics is realted to introception it carries a religious force which is a kind of applied mathematics, but in quite a different sense. In the latter case, Truth is not an incidental notion employed by mathematics, but so largely becomes its soul that the word must be spelled with a capital T."

In the Epilogue of this volume we read:

"There is frequent reference in the book to mathematical analogues. There is a reason for that. The underlying thesis is that the factuality of pure mathematics might be as much in doubt of the factuality of pure metaphysics. But as the factuality of pure mathematics is abundantly proven, there is the presumption that equally well the factuality of pure metaphysics may be proven."

Finally, his view is most fully expressed in the original experiential record[3] under the dateline of October 4, 1936 :

"Once one recognizes the fact that the relative world, or primary universe, is a valid part within the Whole and is relatively real, then the problem of cross-translation from the level of Cosmic Consciousness to that of subject-object consciousness is realized as being of high importance. The possibilities of cross-translation are admittedly limited. The immediate content of the Higher Consciousness cannot be cross-translated, but certain formal properties can be through the use of systematic symbols. In some respects it is like the old problem of the evaluation of irrationals in terms of rational numbers. The ultimate content of the irrationals cannot be given in the form of the rationals, yet, in the radical signs, we have symbols representing the essential unity binding the two sets of numbers. Just so soon as the mathematicians abandoned the effort completely to reduce the irrationals to rational form, and accepted the radical sign as an irreducible symbol of profound meaning, then they did succeed in integrating in their consciousness two quite differently formed domains of reality. This integration meant that

3) Pathways Through to Space (p. 208)

The two domains were found to be logically harmonious, although that which we might call the 'affective' content was discrete. Cross-translation, in something of this sense, is possible with respect to Cosmic and subject-object consciousness. In fact, if the consciousness-equivalents of the entities and operations of pure mathematics were realized, we would find that, in that great science and art, cross-translation in a lofty sense already exists. The Root Source of pure mathematics is the Higher or Transcendent Consciousness, and this is the reason universal conclusions can be drawn with unequivocal validity in pure mathematics. The greater bulk of mathematicians fall short of being Sages or Men of Recognition because their knowledge is not balanced by genuine metaphysical insight. But they do have one-half of the Royal Science. Up to the present, at any rate, the Fountainhead of the other half is to be found mainly in the Orient. The union of these two represents the synthesis of the East and the West, in the highest sense, and is the prerequisite of the development of a culture which will transcend anything the world has known so far."

It is in amplification of the foregoing masterly statement that the present essay is contributed by the editor.

We in modern science do not claim to have more than embryonic ideas about consciousness. In contrast, the traditional East, particularly in its ancient Vedic scriptures, together with the derivative six systems of Indian philosophy, and their formidable Buddhist opponent - the Mahyadamaka, does claim to have a well-developed understanding of consciousness. In the West, by practising "outsight" energetically for 500 years, we have been able to reach a powerful understanding of the outer physical, (and, to some extent, biological) universe. It stands to reason and intuition that the wise men and women of the East by practising insight for thousands of years would have come to a deep understanding of the inner universe of consciousness. Nevertheless, the success resulting from the insistence on a refined analytical and formalizable description of perceptual experiences of the outer world (which is standard in modern science) leads us to expect that these same analytical or mathematical methods may illuminate the subject of consciousness. Particularly relevant to our present inquiry is the analysis of "infinity" - a central recurring theme in the reports of Yogic transcendental experiences as well as the visions of cosmogony in the ancient scriptures. In this essay we discuss infinity as considered in modern mathematics where it has undergone some fascinating developments. The Concept of Infinity in Modern Mathematics There is a statement in the Upanishads[5] which speaks of the two infinities - 1) the universal Supreme Brahman or Self and 2) Creation of the visible universe: when the second emerges from or merges into the first, the first remains the same infinity. This is a remarkable anticipation of the concept of infinity as it is understood today. An infinite set, which we discuss below in more technical terms, has just that characteristic property which no finite set has - that the part can be equivalent to the whole. This means also that, unlike the finite set, no matter how many facsimiles of its parts are added to the whole, the infinite number of elements in the whole is not changed. This is quite opposed to what happens with a finite set.

Let us take an example to engage the imagination. Suppose we have an infinite

5. From the Brihad-Aranyaka Upanishad, Fifth Adhyaya, First Brahmana (also in the Prologue to the Isa-Upanishad, as quoted from the White Yajur Veda, 40th and last chapter):
OM Infinite is that [the supreme Brahman] infinite is this (the conditional Brahman, or the visible universe). From the Infinite (Brahman) proceeds the infinite. [After the realization of the Great Identity or after the cosmic dissolution] when the infinity of the infinite (universe) merges (in the Infinite Brahman), there remains the Infinite (Brahman) alone.
Translation of Swami Nikhilananda The Upanishads, (Bell Publishing Co., New York, 1962)

hotel, with a principle of privacy so that there is only one guest to each room; suppose also that there is one room to each guest so that every room is filled and there are no guests left out. Then a new guest arrives. Now if this were a finite hotel the management would have to say, 'Sorry, try the hotel across the way', but if it is an infinite hotel there is no problem: Install the new guest in Room No. 1, move 'old' guest No. 1 to Room No. 2; move 'old' guest No. 2 to Room No. 3, and so on. No guest is left unroomed and no room is left unoccupied. Now not only can the hotel accommodate one more guest but it can accommodate a million more or infinitely many more. And in fact, if we have infinitely many hotles, each of them infinitely large and all of them occupied, and we decide to dismantle all but one of these hotels, we can put infinitely many infinite hotel populations all into one hotel.

This will be proved later. It is a characteristic property of an infinite set that a part can be equivalent to the whole; and from this follow all the consequences for infinite hotels. As we have said, this property was anticipated in the beautiful saying in the Upanishads in which, in some translations, the word 'fullness' is used to designate what we call 'infinity'.

A little preliminary history of the concept of infinity in the West may be useful. First, most of the ancient Greek thinkers, very clear-headed within certain limitations, abhorred the idea of infinity. With a few exceptions like Archimedes, who partly anticipated the calculus, they abandoned the use of infinity after trying it briefly. There were the famous paradoxes due to Zeno. The most interesting of these paradoxes is about Achilles and the tortoise. Suppose Achilles runs 10 times as fast as the tortoise. Zeno argues, "If you give the tortoise a head start of 10 feet, by the time Achilles covers the 10 feet the tortoise has gone one foot. But then while Achilles covers the remaining foot the tortoise goes another one-tenth of a foot, and so on". Thus, Zeno says, "Achilles never catches up with the tortoise because, each time he covers the distance between him and where the tortoise was, the tortoise has gone another one tenth of the distance".

A paradox is an argument which appears to lead to a contradiction, which is indeed only an apparent contradiction, because it is based on an implicit unwarranted assumption; when this assumption is removed the trouble disappears. The trouble with Zeno's argument became clear with the invention of the calculus and development of the concept of limit. The unwarranted assumption implicit in Zeno's paradox is that the sum of an indefinitely large number of indefinitely small parts is necessarily infinite and cannot in some cases be a finite number. But of course, it can be: add up an indefinitely large number of indefinitely small parts, each one tenth the preceding in magnitude so that they ultimately become indefinitely small, and you get a finite quantity. It is the limit of a sum (a 'series'), where the number of terms goes to infinity. In our case, the series "converges": The limit is a finite quantity. Achilles overtakes the tortoise when he has covered 11 and one-ninth feet.

The coming of the calculus thus changed the picture. Then people became over confident, and they began talking about 'infinitesimals', infinitely small quantities. They also spoke of infinitely large quantities, which would naturally be reciprocals of infinitesimals. But by the beginning of the nineteenth century there was real trouble. Mathematicians added up a series grouping terms one way and got one answer, added it up grouping terms another way and got another answer. There was something wrong with this loose use of infinitesimals. It was always dangerous and sometimes gave clearly wrong answers. At that point, a great revolution took place. It was initiated in 1821 by Augustin Cauchy, who introduced the concept of 'limit'. This was made into a rigorous concept by Karl

Weierstrass in the 1870's systematically and it took the place of the previous loose use of infinitesimals and infinity.

Thus the mathematicians had been properly frightened about infinity and they excluded its use until the last quarter of the nineteenth century. Then a brave man named Georg Cantor, (after the pioneering work of Bernhard Bolzano in 1851) opened up an entirely new outlook. He reintroduced the concept of infinity consistently, through the concept of 'set'. The 'new math' in which many modern children were educated is based on this concept of set.

Cantor's own definition of a set was that 'it is a multitude conceived of by us as a one'. We consider a collection of objects as one population and that makes a set. More picturesquely and explicitly, one mathematician suggests we think of a set as follows: Imagine a transparent tightly closed bag or impenetrable shell. Suppose that all elements of a given collection 'A', and no other objects, are present within the shell. This is a good way to visualize uniting all the elements into the set 'A'.

A set can be either finite or infinite, since it can contain either a finite or an infinite number of objects. Thus the concept of a set provides a very good foundation both for the mathematics of the finite and the mathematics of the infinite. Indeed, among the finite sets are those which contain no elements at all. This 'null set' concept is very important, and reminds us again of India. The people of India are responsible for the invention of the zero, one of the great contributions to human thought. Zero is a number - it is needed to complete the system of numbers. (Subtracting any number ~~for~~ from an equal number leads to zero). More generally, in almost every complete mathematical system there must be a null number. Now there are null sets; some we are sure of, some we are not sure of. An example of a set we are not sure of is the set of all living plesiosaurs. It is possible that the living plesiosaur set has one member - the 'monster' at Loch

Ness. But if that 'monster' turns out not to be a plesiosaur, then that set is a null set.

For an example of the finite case, consider the set of all living people on earth. There is also a set of all living people who have passed their 21st birthday; it is a 'subset' of the first set.

As we have remarked, sets can be infinite too, and that is when the fun begins. One of the first things that Cantor pointed out is that an infinite set has the property that the part can be equivalent to the whole (as with our infinite hotels). Now, how can that be? What do we mean by 'equivalent'?

Cardinality

Even primitive peoples have two concepts of number: 'cardinal' and 'ordinal'. Ordinal number is the kind we are used to on an elementary level (1, 2, 3,...), when we not only encounter a finite set but have all its elements arranged in a definite order. Then ordinal number is based on straight counting. But this system of counting is not practical even for large finite groups (such as people in a big hall). There is, however, another way of measuring the size of a set - finite or infinite - by what is called 'cardinal number' or 'cardinality'. The cardinality concept involves matching between sets without restricting the orders of elements within the sets. We shall first explain the meaning of 'equal cardinality', and later the meaning of 'greater' and 'lesser' cardinality.

Suppose we come into a large hall and are able to have a really good look at everything there. We see many people and many seats, with one person to each seat, and one seat to each person; every seat is occupied and no persons are standing. We see at once that there are just as many persons as seats because they are all paired off. This kind of one-to-one matching is what is usable with an infinite as well as with a finite set. It is the way we measure one infinite set against another infinite set.

The definition of "equal cardinality" or "equivalence" of two sets is then that there is a one-to-one correspondence between the two sets; every member of one set can be matched with a member of the other set, one to one, back and forth, and there are no members left over on either side. It is by this definition that a part can be equivalent to the whole. For example, take all whole numbers 0, 1, 2, 3, etc; take the subset of that set consisting of all even whole numbers 0, 2, 4, 6 etc. There are equivalently many even whole numbers as there are whole numbers, odd and even, because we can match them one to one, back and forth: 1 to 2, 2 to 4, 3 to 6, 4 to 8, 5 to 10, and so on. There are just as many of these as of those, cardinally speaking. Because it goes on forever there is no problem. That is a very important point; it must go on forever; otherwise it would not be true that there are equivalently many even numbers as all whole numbers. Any set with the same cardinality as the set of whole numbers is said to be 'countable'. This is the first infinite or "transfinite" cardinal number.

Cantor showed many other things, such as that there are 'just as many' points on a little piece of straight line as on the whole infinite straight line (i.e., that these two sets of points are of equal cardinality). Even more extraordinary, by this same definition of equal cardinality, there are 'just as many' points on the side of a square as there are in the area of the square. Take a piece of a straight line, say from zero to one, and consider the square one can build on that; Cantor showed that there is a one-to-one correspondence between the points in the square and points on the line. We will not show this explicitly for these sets of points each of which forms a "continuum", but will give the feeling of ~~it~~ (the demonstration) by considering instead an infinite "square lattice" of whole number locations. This will also explain and justify the earlier remark that infinitely many infinite hotel populations can be accommodated in one infinite hotel. This can be done by using one of Cantor's many devices by which he proved a one-to-one correspondence. We arrange it so that every guest in every hotel is labelled by a pair of numbers. The first number (row number) labels the hotel and the second number (column number) labels the room in which the guest is. Thus, the number (h,g) means the (hth hotel, gth room) and therefore is the location of a guest. We will write these all down in a systematic order: (1,1), (1,2), (1,3), etc.; (2,1), (2,2), (2,3), etc.; (3,1), (3,2), (3,3), etc. Thus we can indicate all the guests in all the hotels in a square array of 'ordered pairs'. Now we want to squeeze them all individually and without omissions into the rooms of one hotel which are just labelled by the single sequence of numbers 1, 2, 3, 4, 5, etc. This is done by 'squares'. We start with (1,1) and install him in the first room of the single hotel. Thereafter, beginning with (1,2), we proceed from the top down and then across to the left. In other words, the second installation takes care of (1,2), the third takes care of (2,2); then left to take care of (2,1). Continuing in this manner we take care of all the people, exhausting all the possibilities in the square array and at the same time using up all rooms in the single hotel in a serial order. It is easy to check that nobody is left out, and there is a simple formula for where in the single master hotel each guest goes: Guest g in hotel h goes into the rooms of the master hotel as follows: into room $(g - 1)^2 + h$ if the number h is less than the number g, into room $h^2 + 1 - g$ if the number g is less than the number h. Once we have established a one-to-one correspondence between all guests in all hotels and the rooms in one single hotel we can take care of all of them. So, in short, one can have two - or many - infinities which, added or subtracted, are left infinite.

By letting guest room numbers represent numerators, and hotel numbers denominators, the proof that we have just given can be adapted to show that the set of all 'rational numbers' (fractions, i.e., ratios of two whole numbers) is countable.

One of the things that Cantor found out as soon as he had defined cardinal equality, is that he was naturally led to a definition of what it means for one infinite set to be 'cardinally larger' or 'cardinally smaller' than another. What does it mean for one infinite set to be cardinally larger than another? Suppose I come into an infinite hall and there are infinitely many chairs and infinitely many people, but I see that by some systematic procedure - extrapolated to infinity - I can match all the people to a subset of all chairs but I cannot match all chairs to any subset of all people. In other words, although to every person there is a chair, there is not to every chair a person. (No one is standing but some chairs are empty.) I would immediately say there are fewer persons than chairs. This is how one defines "smaller than". Likewise, there are more chairs than persons. This is how one defines "larger than".

To give an example of a set larger than countable we go to the set of all real numbers. It is easier to visualize their properties if we represent the numbers one-to-one by all the points on a line - the so-called 'real line' - and we associate each point with the number which gives the distance of the point from the origin, zero. For brevity we identify the point with its distance number and we say "the point ½", "the point 1", etc. Some of these numbers can be expressed as bona-fide fractions, i.e., a ratio of two whole numbers (3/4, 2/3, and so on). But many of them cannot, as was found out by the Pythagoreans. These numbers which cannot be expressed as a ratio of two whole numbers are called 'irrational' precisely because they cannot be expressed as a ratio of two whole numbers. (The square root of two is one simple example of such an irrational number. (It is easy to find the point on the line which corresponds to the square root of two: build the square of side-length one, with the real line as its diagonal and with the zero (origin) of the real line at one corner of the square. The diagonally opposite corner to the origin is then the point whose distance from the origin is the square root of two. This follows from Pythagoras' theorem on right triangles. It is easy to prove that this number whose square is 2 cannot be expressed as any fraction.[6] Further, the square root of 2 is a solution of the algebraic equation $x^2 - 2 = 0$. It is an example of what is called an algebraic number, as is also any rational number; the letter is a solution of the simplest algebraic equation $Dx - N = 0$ or $x = N/D$. In general, algebraic numbers are "roots" or solutions of any algebraic equation, i.e., a sum of integer powers of a variable x with integer coefficients - the sum being equal to zero. If we take the set of all such root numbers on the real line between zero and one, or on the infinite real line - it does not matter which - this set is still equal in cardinality to that of the rational numbers and the whole numbers; i.e., the set of algebraic numbers is countable. In fact, this result is a special case of a powerful general theorem:

> If every element in a set can be specified by a finite collection of whole numbers - the 'labelling prescription' or 'numerical indexing' of the set - then the set is either finite or it is countable.

We can state an equivalent theorem, which is even more interesting to the layman, who is more used to words than to numerical indexing. Let us define a set of describable numbers as that collection of numbers such that a precise description in words (this is equivalent to the labelling prescription in the general theorem just cited) can be given of each (and all) of the numbers in the collection.

6. When the Pythagoreans, whose entire world view was based on whole number harmonies, made this discovery they were very impressed. To mark the awesome occasion they sacrificed a hecatomb of oxen, of which Heinrich Heine remarks: "And ever since, when a new truth comes to light, the oxen are very afraid."

In other words; not only is the whole set defined, but each number in the set is distinguishable from every other; not only common but distinguishing characteristics are given by the definition. Then we have that: *any set of describable numbers is countable*; the set of all describable numbers is countable and naturally contains, for example the set of algebraic numbers.

Transcendental Numbers

The set of all algebraic numbers is countable. All those which are not algebraic are called "transcendental". An example of a transcendental number is π, the ratio of the circumference of a circle to the diameter. This is equal to 3·14159. . . indefinitely. It cannot be expressed as any finite decimal, because it cannot be expressed as a ratio of whole numbers. Further, as was proved after Cantor's ~~time~~ early work, π cannot be expressed as a solution of any algebraic equation. It is a transcendental number.

It was a great discovery by Cantor in the latter part of the nineteenth century that the infinity of numbers which cannot be expressed algebraically is a larger infinity than countable i.e., then the infinity of whole numbers. The set of all transcendentals is uncountable in the sense that it outnumbers the set of all whole numbers or algebraic numbers. Even though the algebraic numbers are very dense on the line - indefinitely close to each other everywhere on the line - there are many more numbers on the line which are not algebraic. More generally, the set of all nondescribable numbers (the set of all nondescribable transcendentals) is of a larger infinity than the countable infinity.

The transcendentals outnumber the algebraics in that we can match every algebraic number to a transcendental but we cannot match every transcendental number to an algebraic number. Thus "almost all" the real numbers are transcendental, and they comprise a larger infinity than the whole numbers. A formal proof of this important result may be found in the standard mathematical literature. The proof involves a general idea which leads to the construction of an unending sequence of transfinite cardinals each greater than (but not necessarily <u>next</u> to) the one preceding. This idea may be introduced by a simple example. Suppose we consider a committee constituted of three people. What is the number of ways in which subcommittees might occur? In other words, allowing presences and absences for each of the three distinguishable individuals involved, how many different subcommittees can we have? Clearly each person may be present or absent. This may be combined with the presence and absence of each of the other persons. The upshot is that we have $2^3 = 8$ possibilities, so there can be eight different subcommittees. The cardinal number 8 is of course greater than the cardinal number 3. Formally, we may represent presence of a given individual by the digit 1, and absence by the digit 0, and the number of ways of having a subcommittee amounts to the number of ways in which the digits 1 or 0 may be assigned to three objects, i.e., 2^3, or 8. Alternatively, we can say that we are talking about the number of all possible subsets of a set of three objects, or the number of possible committees in a department of three.

Quite generally, it turns out that we have the following result. Consider a committee A of a given cardinality $\bar{\bar{A}}$. Let $\bar{\bar{S}}$ be the cardinal number of all possible subcommittees of A. (This is the number of all possible subsets of the set A; alternatively expressed, $\bar{\bar{S}}$ is the number of ways in which the digits 1 and 0 may be assigned to each and every member of the set A.) The result is that $\bar{\bar{S}} = 2^{\bar{\bar{A}}}$ is a larger cardinal than $\bar{\bar{A}}$. The proof goes for infinite as well as finite committees (sets). In this manner one knows that there exists an unending ever-increasing sequence of cardinal numbers[7]. Cantor demonstrated that the cardinal number C of the set of all points on a line is equal to the cardinal number of all subsets of whole numbers and therefore C is greater than the cardinal

7. One has often suspected that the committee system is connected with proliferation on to infinity!

number of the set of all whole numbers. His proof that transcendental numbers exist and in a larger infinity than that of whole numbers was given in 1873 and produced a great impression among mathematicians. Cantor had been able to demonstrate the existence of transcendentals without constructing a single concrete example, using only general arguments; but, as the mathematician N. Y. Vilenkin[8)] writes: 'The virtue of Cantor's proof was at the same time its weakness. It was impossible to deduce a rule from Cantor's reasoning which would allow the construction of even a single transcendental number, to say nothing of a test for the transcendence of such numbers as π or $2^{\sqrt{2}}$. His argument constituted, as mathematicians say, a pure existence proof... it became clear that the algebraic numbers met with at every step in mathematics are really extremely rare, while the transcendental numbers so hard to construct were really the common ones.' How hard transcendentals are to construct can be seen from the fact that the French mathematician Liouville had been able, with great effort, to find a few transcendental numbers in 1844. The proof that π was transcendental was first given by Lindemann in 1882. It was a great mathematical event, demonstrating conclusively that it is impossible to 'square the circle', i.e., that it is impossible, with the classical Euclidean instruments of ruler and compass, to construct a square having the same area as any given circle.

The transfinite cardinals are labelled not by Greek, or Roman, or Chinese characters but by the Hebrew alphabet letters. The first one is called Aleph_0, that is, the countable infinity, and the next one is called Aleph_1. There arose the great question: Is the number of points (C for continuum) on a line, or equivalently, the cardinal number of transcendentals, the next transfinite num-

8. Stories About Sets, by N. Ya. Vilenkin, (Academic Press, New York and London 1969).

ber following the cardinal number of integers? Is there an $Aleph_1$ between C and $Aleph_0$? The hypothesis that there is nothing between is technically known as 'the continuum hypothesis' and if it is valid then C is $Aleph_1$.

Some very clever people tried for several generations to confirm or refute the continuum hypothesis. Finally it was proved by K. Godel, whose general studies of the problem of consistency revolutionized formal logic - and P. J. Cohen, that one can either accept (Godel) or reject (Cohen) the continuum hypothesis. One has a consistent theory either way. The situation is similar to that which occurs with non-Euclidean versus Euclidean geometry. It is perfectly consistent either to abandon Euclid's fifth postulate concerning the existence of a unique parallel to a given line through an outside point, or to accept this postulate. In the first case we have the two consistent systems[9)] of non-Euclidean geometry and in the second case we have the consistent system of Euclidean geometry. So in the same way it is possible to have numbers in between $Aleph_0$ and C or not to have such numbers. Thus we are allowed to have different consistent systems of transfinite arithmetic.

It is possible to construct an unending sequence of higher and higher infinite cardinal numbers. In each case we construct a higher cardinal number by considering the number of ways (functions) in which we can assign the values 0 or 1 to each member of the set with the lower transfinite cardinality. This number of ways defined on the set of lower transfinite cardinality gives us a higher transfinite cardinality. Whether or not it is <u>the next</u> transfinite cardinality depends on whether or not one adopts the "generalized continuum

9. In the "hyperbolic" non-Euclidean geometry the fifth postulate is replaced by one admitting an infinite number of parallels in Euclid's sense; in the "elliptic" or "spherical" non-Euclidean geometry there are no Euclidean parallels.

hypothesis". In any case one arrives at an algebra and an arithmetic of the infinite. Here we write only two formulas in the arithmetic of the infinite. One we demonstrated earlier, in the one-to-one correspondence 'by squares' of the ordered pairs to the whole numbers, and the other we have only indicated:

$$\text{Aleph}_0 \times \text{Aleph}_0 = \text{Aleph}_0, \ C = 2^{\text{Aleph}_0}.$$

In order to give meaning to the question raised by the continuum hypothesis we must know what is meant by Aleph_1, "the next" transfinite cardinal after Aleph_0. This was established by Cantor in developing the theory of ordinal numbers into the transfinite domain. The ordinal number of a set results from distinguishing a particular order of the members of the set. The finite ordinals are finite arrangements of the familiar whole numbers. The first transfinite ordinal - labeled ω - corresponds to the set of all positive whole numbers in natural order,

$$1, 2, 3, 4, \ldots$$

and is defined by the following properties:

(a) There is a first member,

(b) There is an immediate successor to every member,

(c) There is an immediate predecessor to every member, except the first,

(d) There is no last member.

If we interchange the words "first" and "last", and simultaneously, the words "successor" and "predecessor", we define $^*\omega$ ("star ω "), the set of all negative whole numbers in natural order:

$$\ldots, -4, -3, -2, -1.$$

Addition of two ordinals is then defined in the obvious way except that order matters when the transfinites are involved. For instance we find

$$\omega + \bar{n} \neq \bar{n} + \omega = \omega$$

($\bar{n}$ is the finite ordinal 1, 2, ...)

These relations hold because on the left side of the unequal sign property (d) is missing, whereas on the right side all four properties, (a) to (d), hold. These two sides are examples of sets (of which there are many other examples) of different order types, which still have the same cardinality Aleph_0. This is because they can be rearranged (changing order!) to be in one-to-one correspondence with the set of all whole numbers, i.e., they are countable. All such sets are called <u>well ordered</u> if two sets of conditions are satisfied. First: they are <u>simply ordered</u>, i.e., distinguishably and unambiguously ordered and also linearly (if A precedes B and B precedes C, then A precedes C.). Second: they satisfy the following three properties:

a) There is a first member

b) There is an immediate successor to every member except the last <u>if there is a last</u>.

c) Every "fundamental segment" of the set - i.e., any lower segment with no last member - has a "limit" in the set; the "limit" is the member next following all members of the fundamental segment.

(In the example

$$\omega + \bar{n} = \omega, 1, 2, \ldots n$$

n is the last member, ω is a fundamental segment, and 1 is the limit of ω .)

Products and powers of ordinals were defined by Cantor. (Again "order" is of the essence and must be preserved carefully) and it turns out that

$$\omega^n r + \omega^{n-1} s + \ldots \omega, \ldots \omega^{\omega}, 111\ \omega^{\omega^{\omega}} \ldots$$

which are the ordinal numbers of well-ordered sets all have the same cardinality Aleph_0 since all can be arranged to be in one-to-one correspondence with the natural numbers 1, 2, 3, ... Now the class of all possible kinds of well ordered sets each of cardinality Aleph$_0$ defines a set with transfinite ordinal larger than that of any of the constituent sets. This transfinite ordinal is represented by Ω and the corresponding cardinal number of this entire class was called Aleph$_1$ by Cantor. Similarly the class of all possible types of well ordered sets of cardinality Aleph$_\nu$ forms a (ν+ 1)st class whose cardinality Cantor designated by Aleph$_{\nu+1}$ and so on. Cantor thus obtained an infinite sequence of ever-increasing transfinite cardinal numbers. But, as Cantor writes: "Even this does not exhaust the conception of transfinite cardinal number. We will prove the existence of a cardinal number which we denote by Aleph$_\omega$ and which shows itself to be the next greater to all the numbers Aleph$_\nu$; out of it proceeds in the same way as Aleph$_1$ out of Aleph a next greater Aleph$_{\omega+1}$, and so on, without end."

Now we understand what the issue of the continuum problem and its generalized form means. In fact the question is deeply related to the question of what order means: "The question whether every transfinite cardinal number is necessarily an Aleph-number... is equivalent to the question whether every (set) is capable of being (well-ordered)." 11)

The further development of the theory of sets is very interesting. It led to a revolution in mathematics because if provided a basis for both the mathematics of the finite and the infinite. One finds the theory these days in all kinds of books, some still called 'set theory', others 'measure theory', others 'theory of real variables', and so on. All the great branches of modern mathematics - functional analysis, topology, higher algebra - have set theory at their foundation. They in turn have many applications in modern science and technology. Even in that grand old branch of mathematics, geometry, concepts which were originally

10. Contributions to the Founding of the Theory of Transfinite Numbers by Georg Cantor (translation and introduction by Philip E. B. Jourdain, Dover, N. Y. 1955) p. 109.

11. E. W. Hobson The Theory of Functions of a Real Variable (Cambridge University Press)p.238.

taken for granted, such as 'curve', 'surface' and 'volume', have been revised. Propositions which everybody had thought were obvious, such as that a square is two-dimensional, a cube three-dimensional, and so on, had to be re-examined, and very strange results were found by the mathematicians. Using Cantor's definitions, they found all kinds of new and bizarre mathematical objects coming into the mathematical zoo. For example, they found that there are infinitely prickly curves and also curves which have non-zero areas. If we define a curve as carefully as we can by a significant definition, then there are curves which have a finite non-zero area. It was once thought that a curve could only have zero area, but there are curves which are so complex and cover so much of a plane that they have a well defined positive area. In contrast, there are domains which look two-dimensional - which look like surfaces - but which have no well defined area. This can come about because we are dealing with a region for which the boundary curve turns out to have non-zero area. Therefore, if one adds the boundary to the region one has a larger area; if one takes it away one gets a smaller area. All kinds of strange properties like this have emerged, and the mathematicians have become very careful in their definitions and very strict in their arguments.

And so, too should we be in all formal matters. As we know, however, there are aspects of Reality which are not formalizable, and here the door must be left open to the intuition and insight by which alone the unknown may be experienced. The fact that almost all numbers such as the set of transcendentals, finite though each may be, require an infinite expression in terms of the natural numbers evokes a resonance with the deeper levels of consciousness. Likewise the existence of an unending sequence of ever higher infinities is an intimation to us of the existence of higher levels of consciousness.

Referring back to the discovery of <u>alternative</u> <u>different</u> <u>systems</u> of transfinite arithmetic one can hardly overestimate the importance of the discovery that "both" possibilities are true in such instances but - let it be noted - not in conjunctive affirmation, but what may be called disjunctive or complementary affirmation: There exists more than one geometry, more than one algebra, more than one transfinite arithmetic etc. This is a good place to refer to the devastating tetralemma dialectic (Catuskoti) of the great Buddhist anti-logician Nagarjuna (1st-2nd century A.D.). The structure of this four-cornered negation which Nagarjuna employed to knock down (in good mathematical spirit by demonstrating internal contradictions) the arguments of those who attempt to analyze reality logically is as follows: There are four alternatives:

(i) A positive thesis

(ii) The opposite counter thesis

(iii) A conjunctive affirmation of the first two

(iv) A disjunctive denial of the first two

Clearly the fruitful development of mathematics shows the possibility of another alternative, modifying number (iii): both thesis and counterthesis are true but in different systems, each self-consistent in itself. Thus might be resolved the remaining differences - subtler and lesser than the protagonists of each may have maintained - between Sankara's Advaita Vedanta and Nagarjuna's Madhyamaka Buddhism. Then, the indescribable and translogical fullness of Brahman and absolute subjectivity of Nirvana are complementary and equivalent.

Again, an informal response is evoked by the extraordinary theorem in formal logic which Gödel proved: in every sufficiently rich formal logical system there exist unformalizable elements which can neither be proved nor disproved. Such statements are called 'undecidable statements'. (Echos of Buddha and Nagarjuna!).

This theorem means that there exist unformalizable elements in every sufficiently rich formal system. It is not surprising that Godel's discovery is regarded as one of the most profound of our time. It is a tremendous revelation to modern man that formal logic can prove its own limitations. One feels that this discovery is connected in a deep way with the principle which appears over and over again in Vedic science and in the entire Eastern traditional world view: the principle of inclusion or co-existence of opposites or, sometimes, the co-nonexistence of opposites. For example, the great Hymn of Creation in the Rig Veda begins: 'There was not non-existence, there was not existence at that time...' One does indeed feel that an ancient door has been opened again by such theorems as Godel's theorem. There is some profound nuance of meaning in that the principle of inclusion of opposites makes its appearance centrally in the ordering of those very infinities which provide such an evocative parallel to higher states of consciousness.