A Mathematical Supplement to the Lectures of Franklin Merrell-Wolff[1]

Joseph Rowe

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Joseph Rowe

> In mathematics we excel all other cultures, and as I see it, all other genuine superiority we may have resulted from this mathematical excellence. In other respects, as far as the greater and more durable values are concerned, there are other cultures in the Orient, whether of the present or the past, that just as clearly excel ours. Now, it is by its power, and not its weakness, that an individual or a class attains the best. Thus, I would select the mathematical road as the one of preeminent power so far as western culture is concerned.[2]
>
> Franklin Merrell-Wolff

Introduction

Every student of the lectures and writings of Franklin Merrell-Wolff must, sooner or later, be struck by the current of pure mathematical thought, which runs like an underground river through much of his work, emerging here and there into explicit references and even technical diagrams, particularly in some of the audio-taped lectures. Yet there are many earnest and capable students and readers of this work whose attention begins to fail just at this point. This is an unfortunate situation, and rather ironic in light of the above quotation. It is also quite unnecessary, in the opinion of the present writer.

If pure mathematics is of the very essence of the power of our Western heritage, then how does it come to be that so many have become estranged from this heritage? There is a blame to be laid here, and it is at the feet of our educational system, which generally produces mathematics teachers, particularly at the psychologically crucial secondary levels, who have little knowledge of, or feeling for, the deeper significance of the great developments in mathematics, and even less feeling for the aesthetic beauty of mathematics. This is not a new situation. For too many years, in both Europe and America, mathematics has traditionally been presented as a dry and forbidding subject, a test of cleverness and facility of abstract manipulation, so that all but those with a particular talent in this area are quite likely to come to regard this study as a joyless chore, and the sooner completed the better.

But in fact, one need have no particular talent in this area in order to develop a deep appreciation of the significance and beauty of mathematics, any more than one needs to have a special musical talent in order to develop a deep and informed appreciation of music. In fact, it is not an uncommon occurrence to find genuinely talented individuals who themselves are quite defective in their ability to appreciate the deeper significance of mathematics, even though it be their chosen profession. The tragedy of this cultural complex might be illustrated by stretching the above analogy somewhat, and imagining a culture in which the subject of music is frequently taught, even at advanced levels, by very unmusical professionals, some of whom are even tone-deaf, but who laboriously and assiduously present to students all the theory of scales, harmony, counterpoint, etc., examining, analyzing, and expostulating upon

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[2] Franklin Merrell-Wolff, *The Philosophy of Consciousness Without an Object* (New York: Julian Press, 1973), 172.

written scores of compositions, classifying and annotating them, and all the while rarely or never actually playing with real feeling, or even playing recordings of musicians who can perform with real "musicality." To a member of this imaginary culture who had nevertheless managed to both get through the difficult courses, and play and play and hear some real music as well, the general situation could only appear as very unfortunate, to say the !east. And so it does to his mathematical analogue in this world. Why this situation has care about with mathematics remains a mystery, though

The situation was not always thus. In the Greek world, sometime around 500 B.C., when Western mathematics was being born, it was taught as one of the most sublime of disciplines, indispensible to Philosophy, and deeply connected to the Mysteries. This is very far indeed from the current situation. Although there is some (often rather half-hearted) admission by modern thinkers of the profound connection between mathematics and philosophy, any suggestion that certain discoveries of pure mathematics can be significant in considering such problems as the relation between ordinary subject-object consciousness and the transcendental consciousness would be greeted in most quarters with derision or shrugs, if only because the possibility of a transcendental consciousness has long been rejected by most professional philosophers and mathematicians.

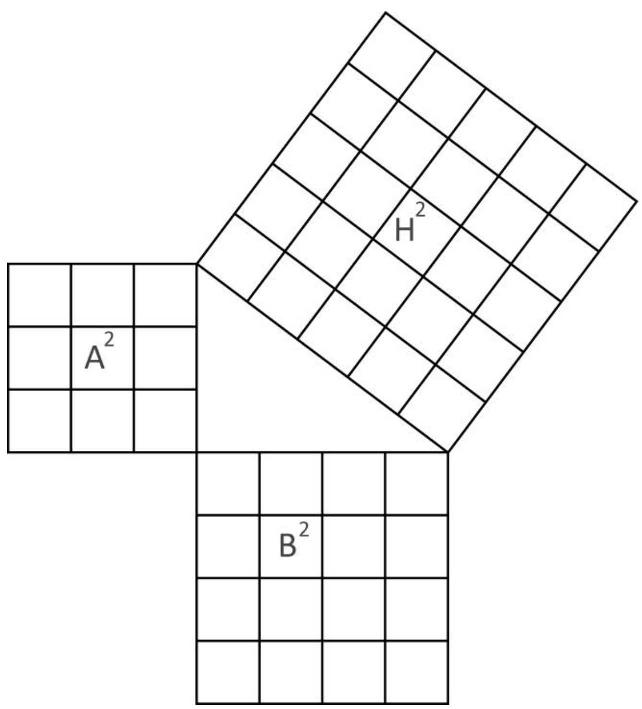
A very important characteristic of the pioneering nature of Merrell-Wolff's work is his discovery of the profound relevance of certain ideas from pure mathematics as *symbols* of the relation between ordinary consciousness and transcendental consciousness. There have been other mystics who have attempted to rediscover the original dimension of mathematics as a language of the Mysteries, but few if any of these have combined this desire with a thorough training in modern mathematics, especially the mathematics of the last 150 years. It would appear that Merrell-Wolff is virtually alone among modern mystics in this, and to one who can appreciate the technical aspects of these mathematical references (the last one hundred years turn out to be especially important here), the connections are extremely beautiful and significant. It is hoped that this supplement will at least put some readers on the road to sharing this experience, and thereby reclaim some of the essence of their wonderful Western heritage, regardless of the culture of origin—this heritage of Pythagoras is, in a vital sense, open to all, not just to specialists.[3]

[3] Because of practical considerations, I have had to write what follows under the general assumption that the reader retains a fairly good command of the material covered in a fairly good high school course of mathematics. If you are rusty in this, you might read over some textbooks on secondary school algebra, geometry, and elementary trigonometry; or better yet, get together with a friend who does retain same command of this material.

I. The Pythagorean Theorem and the Axiomatic Method

If we wish to settle upon one central figure in that crucial development of thought which took place in the Greek world around 500 B.C., the best candidate would be Pythagoras. Among the many remarkable stories about this individual, it is said that he was initiated into the Egyptian Mysteries. After leaving Egypt, he began to teach a system which, apparently, was an entirely new synthesis of Egyptian, Greek, and perhaps other streams. It is known that mathematics was an essential element of the Pythagorean School, and a theorem was announced, which has come to bear his name. This famous theorem says that, given any right triangle, the square on the hypotenuse is equal to the sum of the squares on the lesser sides:

For any right triangle whatsoever, with sides A and B, and hypotenuse H,
$H^2 = A^2 + B^2$ (hence, $H = \sqrt{A^2 + B^2}$).

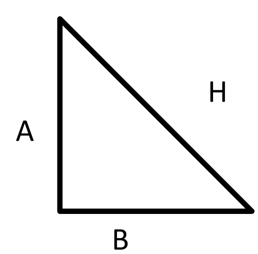


The practical importance of this principle can hardly be exaggerated. Carpenters, surveyors, scientists, technicians, and many others have come to use this principle so frequently and universally, that our technical civilization, whether of the East or West, would be unthinkable without it. And yet, as Merrell- Wolff points out, it is very important to understand that it is not at all this principle in itself which makes the achievement of Pythagoras and the later Greeks so remarkable. In fact, there is ample evidence that the principle itself was widely known before, in Egypt and Chaldea, and even reportedly in China. What was utterly unique with Pythagoras and the Greeks in general, was the idea that it is *not enough* to accept such a principle just because it works, because it always holds true to the test of experience. To that early flowering genius of the Greek mind, such a principle could never be satisfying until it was shown to be a necessary logical consequence of more fundamental propositions. Thus was born the concept of proof, or demonstration, and what we now call the axiomatic method.

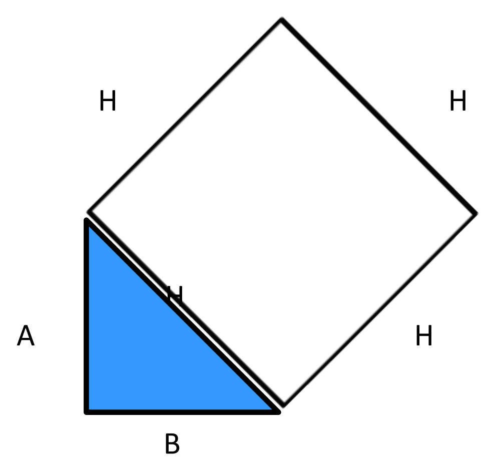
Pythagoras reputedly did succeed in such a demonstration of the theorem which bears his name, and this was one of the great early triumphs of the Greek mind, lying at the very source of what we call Western Civilization. We shall present a demonstration of the theorem here. It is far beyond the scope of this paper to rigorously list the more fundamental assumptions, or axioms, which are used in such a demonstration, and so we assume on the part of the reader just enough of a vague memory of high-school geometry so as to appreciate that this is indeed a valid proof, i .e., an argument which is based on the axioms, which we now call Euclidean. Well over a hundred different proofs of this theorem have been discovered since Pythagoras. The one we now present is one of the simplest and most elegant, and at least one writer attributes its ultimate origin to Pythagoras himself!

Proof:

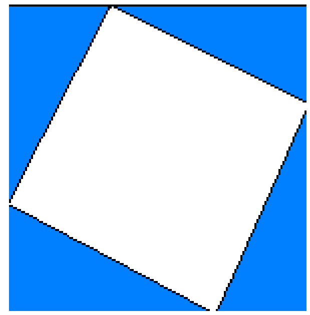
1.) Given any right triangle:



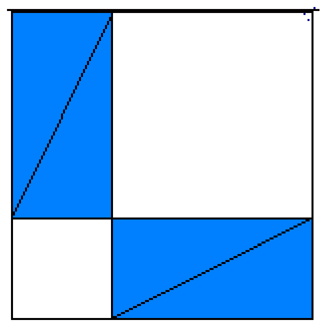
2.) Form the square on the hypotenuse:



3.) Now, construct a large square as follows:



4.) Finally, rearrange the four triangles inside the large square:



The two white squares in this large square are each squares on the legs A and B of the triangle. Since their area is that of the large square minus the four triangles, we know it must equal the area of the white square in (3). Therefore, the square on the hypotenuse (the inside square of (3)) has an area equal to the area of the sum of the squares on the legs.

This is far from being a "rigorous" proof. However, a little contemplation will establish that it is sufficient to enable any reader to construct such a proof, either using Euclid or the modern system of axioms.

Since the time of Pythagoras and Euclid, the axiomatic method has undergone considerable refinement and sophistication, particularly in the last hundred years, and it is to the modern account of this method that we shall refer here as its terminology and organization is superior to that of, say, Euclid, in its rigor, simplicity and elegance . The axiomatic method is the key to the power of Western thought, and is the single characteristic which differentiates it from all non-Western thought we know of. It has, through the ages, seeped into the unconscious thought processes of virtually all who have undergone a Western-influenced education, whether living in what is generally considered to be the Occidental world or not.

Let us take one step toward making this process more conscious, and list the four basic ingredients which constitute an *axiomatic system*: (1) a small number of *undefined*, or "primitive" terms, peculiar to the system (some examples of undefined terms from geometry are: "point," "line," ''between"); (2) a certain number of *definitions*, or new terms, which are clearly and logically defined by means of the undefined terms (an example from geometry: The statement that AB is a "segment," means that there is a point A, and a point B, distinct from A, such that the segment AB is the collection of all points between A and B); (3) a certain number of fundamental, unproven statements, which use the vocabulary of the undefined and defined terms (these of course are the *axioms*, and an example from geometry is the statement: Every line is a collection of points); and (4) finally, a number of statements, quite large in most systems, which use the vocabulary of the undefined and the defined terms, and are *proven*, by a rigorous logical argument, to follow from the axioms. The discovery and proof of the body of *theorems* is the main work of the mathematician. In the course of most arguments, a previously *proven* theorem is referred to "as if" it were an axiom, for the sake of economy and simplicity. We shall not go into the role of exactly what "logic" means here. Let us assume, as did the Greeks, that "logic" is a process of reason, evident to any thinking being, which makes use of a language which is fundamentally clear and universally acceptable. Such words as "all," "some" "any," "if . . . then," "collection," "set," etc. are considered as part of this logical language. This viewpoint would certainly be considered naive to modern logicians and mathematicians, but there is no harm in accepting this viewpoint for the moment.

An axiom system is considered to consist of all of these ingredients, plus the arguments which demonstrate the theorems. Notice that there are very few restrictions as to what constitutes a valid axiom system. A moment's thought should suffice to convince one of the following conditions, not explicitly stated above: no valid axiom system should be allowed to contradict itself; i.e., no axiom system can be allowed in which it is possible to derive two contradictory statements. Furthermore, no axiom must be derivable from the other axioms (if it were, it should be reclassified as a theorem). Also, the total number of axioms must be finite. With these natural and reasonable restrictions added, we see that many, many types of systems are possible. Two valid axiom systems may utterly contradict each other, or they may share vocabulary terms, such as "point," and yet use these terms in radically different and incompatible ways. No judgment is made as to which of two differing systems is "true," as long as each is *consistent*, in the sense of being free from inner contradiction. The number of possible valid systems is thus infinite. Different systems may be related to each other in a variety of ways. They may contradict each other, complement each other, or both, or they may be apparently unrelated. Also, one system may be a sub-system of a larger system. The undefined terms of a system are not required to refer to any sense-experience, although they in fact do so. Furthermore, there may be more than one *interpretation* of the undefined terms of a given axiom system (referring to entities outside that system), each of which is entirely consistent with the system, but having very different meanings. Generally, mathematicians are interested in systems which are "fruitful," "significant," ''beautiful," or simply "interesting." Such terms contain a subjective value element, and different epochs may have different ideas about just what is "interesting"—and yet virtually all mathematicians would agree that certain systems are *trivial*; for example, a system using the words "point" and "line," in which it turns out that there is only one "point" and only one "line" containing it, would certainly not be either fruitful or interesting, though it is no less valid than the Euclidean system, which gives us the familiar, and very interesting, two-dimensional plane.

An example of an axiomatic system follows. Actually, this is sort of a mini-system—more precisely, a sub-system of the system called Arithmetic, or the Real Number Continuum. If you find what follows too abstract for your taste, drop it for now and come back later. A sense of rigor, and feeling for the austere beauty of mathematical formalism is not developed overnight, usually.

Undefined terms: number, sum, negative

Axiom I: If x is a number and y is a number, then there is one and only one number, called the *sum* of x and y.

Definition 1: 'x + y' means the sum of x and y. This association is called "addition."

Definition 2: The symbol '=' means "is."

Axiom II: If each of x, y, and z, is a number, then (x + y) + z = x + (y + z).[4]

Axiom III: There exists a number, called zero, or 0, with the following property: for every number x, 0 + x = x.

Axiom IV: If x is a number, then there exists a number y, called the negative of x, such that x + y = 0.

Definition 3: The negative of a number x, is symbolized as: -x. Hence x + (-x) = 0.

Axiom V: If x and y are numbers, then x + y = y + x.

This is all we know. Questions and observations about these axioms may lead to theorems. We are supposed to know no more about numbers, sums, or negatives than the above statements tell us. Now, look at Axiom III: it does *not* tell us that there is *only* one number with the "zero-property"; i.e., that when added to any given number, the sum is that number. Is zero unique in having this property? This question and some penetrating reflection on the axioms, leads to this theorem:

Theorem 1: Zero (0) is the only number having the zero-property, i.e., such that for any number x, 0 + x = x.

Proof: We begin by *supposing* that z is a number such that z + x = x, for any number x, and then show that z can be none other than 0.

Let x be any number:

1) Then z + x = x. (Hypothesis)
2) There exists a number -x. (Axiom IV)

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[4] The () sign does not really rate a formal definition, since it is merely being used, as in normal English, to isolate a word or clause and save a more tedious formulation without using ().

3) There exists only one number (z + x) + (-x). (Axiom I)
4) (z + x)+ (-x) = z + (x + (-x)) (Axiom II)
5) Since z + x = x, by hypothesis, and since this sum is unique by Axiom I, then we may substitute x for (z + x) in expression (4), which gives us: x + (-x) = z + (x + (-x)).
6) But x + (-x) = 0, by Axiom IV, hence, by the uniqueness of sums in Axiom I, we can substitute 0 in the above expression: 0 = z + 0.
7) Now, z + 0 = 0 + z, by Axiom V; then, applying Axiom IV, z + 0 = 0 + z = z substituting z in the above expression 6), we have: 0 = z.

End of proof.

Theorem 2: (Uniqueness of negatives): If x is a number, and n is a number such that x + n = 0, then n can be none other than -x.

Proof:

1) Given any x and n, suppose that x + n = 0. (Hypothesis)
2) There is a number -x. (Axiom IV)
3) Then, applying Axiom I, (-x) + (x + n) is a unique sum; furthermore, from 1), this sum must be the same as (-x) + 0; i.e., (-x) + (x + n) = (-x) + 0.
4) Now, (-x) + (x + n) = ((-x) + x) + n. (Axiom II)

We leave the reminder of the proof as an exercise.

Note that we have not used the word "subtraction." In fact, it is a logically superfluous notion in the axioms of the real number system. However, it is a very convenient notion. A little thought should enable us to concoct a suitable definition:

Definition: The statement that y is the *difference* of b from a, or that b *subtracted* from a is y, or that y = a - b, all equivalent terms, means that: y = a + (-b).

Theorem 3: If y is a number and z is a number, then there is only one number whose sum with y is z; and this number is the number z – y.

Theorem 4: If x is a number, then - (-x) is x.

Theorem 5: -0 = 0.

We the leave the proofs of these theorems as exercises.

What sort of a system have we got here? Notice that the nature of this system is completely independent of any previous associations we may have with the undefined terms number, sum, and negative. It would remain *essentially the same system*, if we substituted "tove" for number, "wabe" for sum, and "glitch" for negative. Axiom I might then look like this:

Axiom I': If x is a tove and y is a tove, then there is only one tove, called the wabe of x and y, and written x + y.

If we were to go on in this vein, we would have a system perfectly equivalent to the one we have been developing, and we could prove the same theorems about it. But naturally, we want to ask the question: Can we *interpret* this formal system as being the familiar number system we all know?

A little reflection will show that this system is far too terse to be an adequate axiomatic basis for the number system. There is no way to generate anything analogous to multiplication, for one thing. Furthermore, a finite model, or interpretation, would satisfy this system, whereas the number system we are familiar with is infinite. Some more axioms, and some further undefined terms would be needed in order to have an axiom system which could serve as a basis for our familiar number system. However, this system is clearly a *sub-system* of our number system: every one of the axioms so far given would have to be true in the familiar number system. Then why did we stop with the axiom of addition? Other than practical reasons of space, it turns out that this mini-system is an interesting thing in itself. Any set of objects, with some "operation," say *, as in x * y = z, which satisfies the axioms we have given is called a group, and our number system is but one of many possible groups. In fact, there are some sets with operations, still called groups, for which Axiom V is not true, i.e., x * y ≠ y * x.

Thus, many different models, or interpretations, are possible for any given axiom system, without affecting the essential structure of the system; and many different axiom systems are also possible. It was probably while thinking of this fact, that Bertrand Russell coined his famous witticism: "Mathematics is that discipline in which we do not know what we are talking about, nor whether what we are saying is true."

II. Reality is Inversely Proportional to Appearance: $R = 1/A$

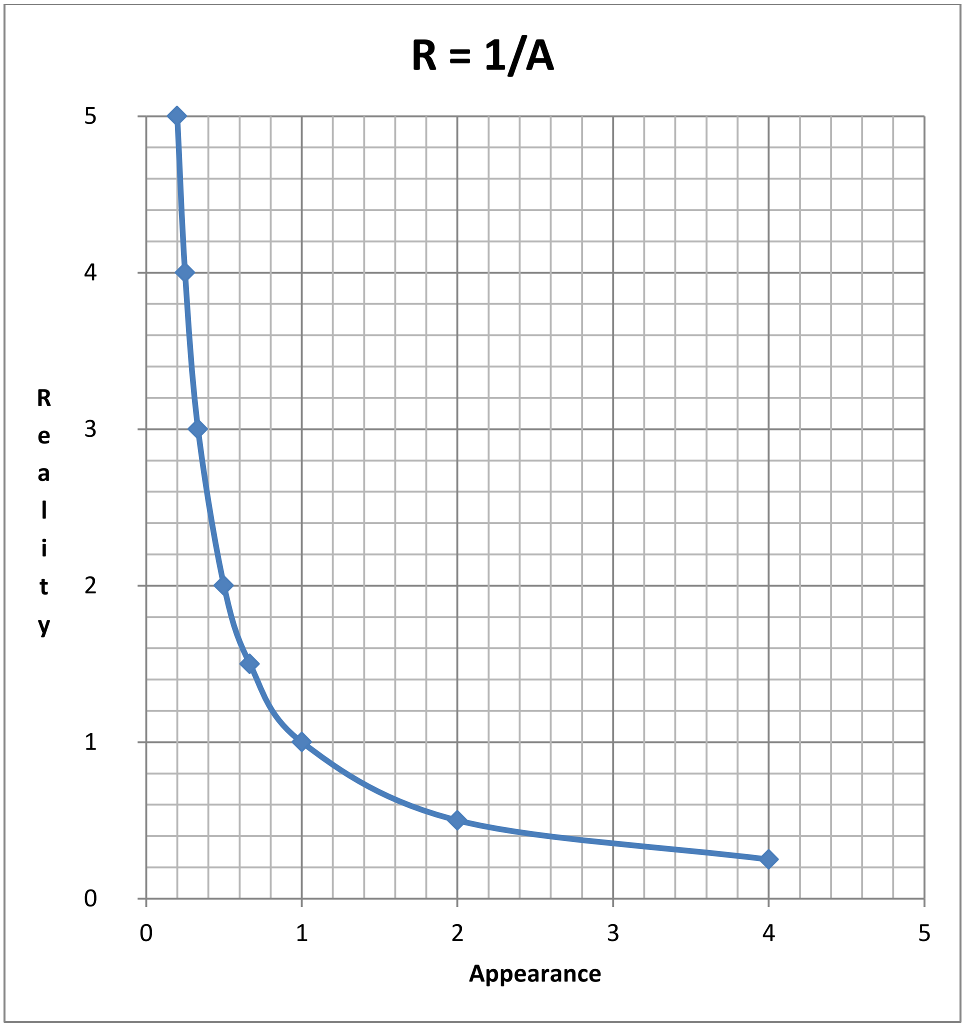
This algebraic formula is a good example of Merrell-Wolff's thesis that pure mathematics is a veritable unmined treasure of symbolic material, which can be used in new ways to suggest Transcendent consciousness. As well as being significantly related to the mandala which appears on the cover of *The Philosophy of Consciousness Without an Object*, this equation has a very interesting symbolic relationship to the important breakthrough in his thinking, which, as he tells us in *Pathways Through to Space*, served to clear the way for the Realization which later occurred.

In this equation, we have two variables. R stands for Reality, and A for Appearance. This relation, R = 1/A, is called inverse proportion. Notice what happens as we choose a variety of numerical values for A: as A assumes values larger than 1, R begins to decrease rapidly. At A = 10, R becomes 1/10; at A = 1,000, R becomes .001, etc. On the other hand, if A = 1, then R = 1 also. Hence the point at which R = 1 and A = 1, is a kind of dividing line, or balance point. When A diminishes to less than 1, R becomes larger. For example, if A = .00001, then R = 100,000. And on and on, without limit in either direction, either R or A is permitted to grow without restriction, and to decrease without restriction, save one: at no point can either R or A assume the value of zero, for the operation of dividing by zero is considered to be undefined, and therefore meaningless, in the normal axiomatic system of numbers.

Let us form a geometric picture of this relationship by making a table of selected values of R and A, and plotting the resulting figure on Cartesian graph paper. Notice that negative values are also theoretically possible for R and A; however, we shall consider only the positive values to begin with.

A	**1/5**	**1/4**	**1/3**	**1/2**	**2/3**	**1**	**2**	**4**
R	**5**	**4**	**3**	**2**	**1 1/2**	**1**	**1/2**	**1/4**

And now, taking R as the vertical axis, and A as the horizontal axis, we plot each of the values determined above, with the resulting figure:

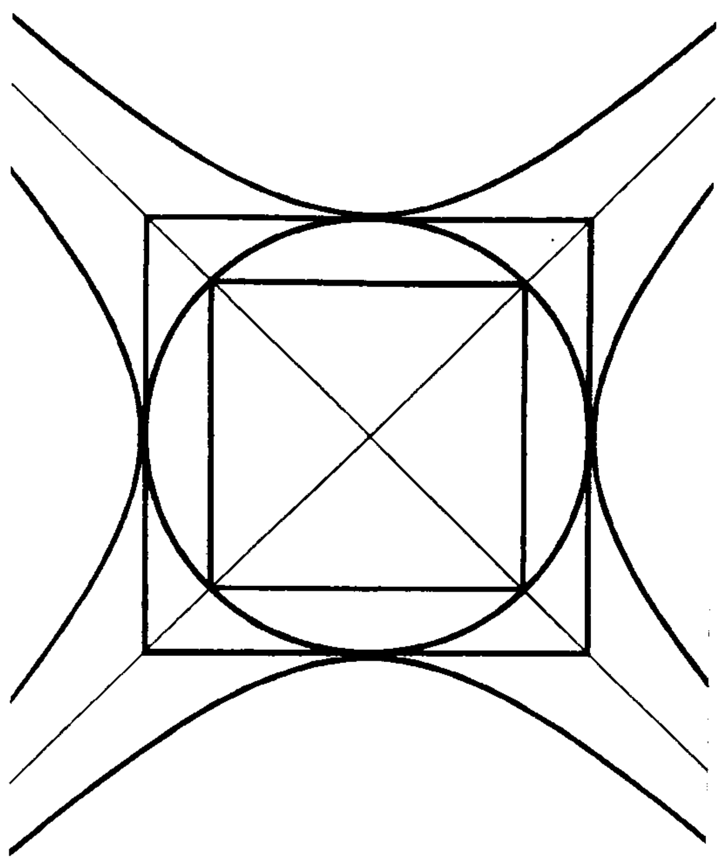


Let us contemplate this figure—neither the draftsmanship nor the accuracy is very important—after all, this curve is not an object of the sense world in any case. Notice that it is infinite in extent, and yet never attains the vertical or the horizontal axes. As R increases without limit, passing through as huge a value as one may care to choose (it may attain, if you like, a number greater than the entire number of electrons in this galaxy, and pass infinitely beyond that too, without yet touching the vertical axis at which A would be zero), A simultaneously must decrease without limit; and conversely, as A assumes equally huge values, R decreases without limit, attaining any arbitrarily minuscule value, and yet without the curve attaining the horizontal axis. Here, then, we have a graphical representation of Merrell-Wolff's realization that "Reality is inversely proportional to Appearance": the more intensely consciousness is focused in the world of appearances (or A assuming large values), the smaller the value of Reality; and the less intensely consciousness is focused on the world of appearances (or A assuming small values), the larger the value of Reality.

Perhaps you have noticed the similarity of this curve to the hyperbolas of the mandala that Merrell-Wolff uses. As an exercise to the reader, take a piece of graph paper, and plot the shape of R = 1/A, as we have done; and then, let A assume these same values, but with a negative sign (i.e., A = -¼, -⅛, etc.), find the corresponding R values, and plot these points on the *same* piece of graph paper. When this is done, still using the same piece of paper, plot the complete curve of the following equation: R = -1/A.

This will look still more like the mandala. In order to get the final form, Merrell-Wolff has performed other operations on the figure (including a rotation of the entire figure through an angle of 45°), for reasons which we won't go into here, except to say that they would appear to be both philosophical and aesthetical.

In the final figure, he has added an inner square, which does not have any point in common with the hyperbolas, a circle which perfectly circumscribes the small square, and is itself tangent to the hyperbolas at four points, and a larger square, which perfectly circumscribes the circle, and is itself tangent to the hyperbolas at the same four points. And, circumscribing all of these figures is the infinite curve of the hyperbolas. This is a very beautiful and profound figure, which invites long contemplation.



We certainly have not gone into all of its symbolic meanings here. We close this commentary here with a paraphrase of Merrell-Wolff's own description of the symbolism of these figures, representing the following phases of consciousness:

- The inside square: complete determination by a finite process; ordinary consciousness.

- The circle: that which lies beyond conceptuality.

- The outer square - a subtler conceptuality, rendering thinkable what was unthinkable; it is bound, not by the circle, but by

- The hyperbola-figure: embracing infinity.

III. The Symbol of the Transfinite

> When the finite is placed in relation to the infinite, as one can plainly see everything happens. If it (the finite) comes first, it goes into the infinite and disappears there. If however, it knows this, and takes its place after the infinite, then it remains preserved and joins itself to a new and modified infinity.[5]
>
> Georg Cantor

The idea of the infinite has exercised a unique fascination for the human mind since time immemorial. In religion, art, philosophy, and mathematics, it turns up over and over again, always carrying with it the aura of mystery. It is curious and worthy of note, that our language compels us to speak of it in an essentially negative way: infinite—"non-finite," "that which is not finite."

The confrontation with the infinite in mathematics began with the Greeks, and caused them a good deal of perplexity. The paradoxes of Zeno invoked the idea of the infinite and arrived at such disturbing "demonstrations"—such as the conclusion that there can be no motion, or that if there is motion, then Achilles cannot overtake the tortoise—these and other results of the invocation of the infinite were like a crack or a flaw in the magnificent edifice of pure reason which constituted their system of mathematics. For centuries these nagging problems were simply laid aside. The second great encounter with the infinite in mathematics took place in the latter part of the 19th century. For the first time, a powerful new conceptual framework was discovered, which enabled mathematicians to dispose of the Zeno paradoxes once and for all, and to incorporate the "irrational" numbers which so disturbed the Pythagoreans, into a grand axiomatic framework in which the infinite occupied a central position.

This new conceptual framework was the theory of sets and its applications, and was pioneered by such original and creative thinkers as Bolzano, Weierstrass, Dedekind, Zermelo, and above all, Georg Cantor (1845-1918), a German-Jewish mathematician whose startling discoveries about infinite sets both founded a new age of mathematical thought, and aroused a violent controversy which continues today in many mathematical circles. This controversy even went so far as to involve social persecution of Cantor by certain of his German colleagues, arousing the shameful and ugly specter of anti-Semitism.

This is all the more surprising in light of the fact that the Theory of Sets is now regarded by most mathematicians as the very foundation of all mathematics, and the common meeting ground of logic and mathematics. To combat this idea and to further discredit the ideas of Cantor (for which he had a violent dislike), the eminent mathematician Kronecker coined the aphorism: "God created the counting numbers (i.e., the positive integers), all else is man's handiwork." However well this may ring literarily, it would appear that this is simply not true, at least from an observation of how mathematical thinking arises in developmental stages of children, and in the evolution of cultures. No observant parent has to read Piaget, for example,

[5] Apropos of a discussion of transfinite ordinals, and in particular, the result that, whereas $1 + \omega = \omega$, it turns out that $\omega + 1 \neq \omega$.

to notice that a child must first learn to form a concept of set, a concept of disparate objects grouped together as a whole, before it can learn to count at all. The concept of set is prior that of counting and numbers, although it tends to remain "behind the scenes" in most people's thought, so much taken for granted that it is rarely brought forth as a possible object of conscious scrutiny, for the very reason that it forms the natural basis of virtually every form of conscious reasoning.

It is probably for this very reason that a theory of sets arose so late in the history of mathematics. To further illustrate the fundamental nature of sets and their operations, we take an amusing example from George Gamow. It is said that the Hottentots are so primitive that they have no concept of counting number beyond three (never mind whether this is anthropologically true or not!). When counting a group of objects, they say 1, 2, 3 . . . and "many." Now, suppose that a certain wealthy and intelligent Hottentot has a group of cattle which he is putting out to pasture, in the care of a herdsman whom he suspects of stealing a cow or two occasionally. Since he can't count beyond 3, he cannot use counting to check on the herdsman. However, he hits upon the following scheme: gathering together a large pile of rocks, he takes one and only one rock successively from the pile, and ties it to each cow's tail before they are led out. He then discards the remainder of the pile, and removes each rock from the tails, and keeps the rocks together in a safe place. When the cows are led back in at the end of the day, he then takes his group of rocks, and affixes each one to the tail of a cow: if any rocks are left over, he knows a cow is missing; on the other hand, if there are not enough rocks, he knows that extra cattle have somehow been acquired—and all this without knowing how to count. A modern mathematician would say that he has effected a reversible transformation, or a one-to-one napping, of a certain set of rocks, onto a certain set of cattle.

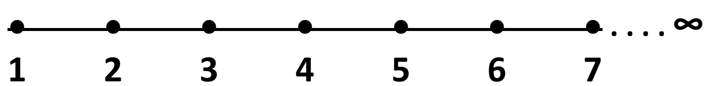
This idea of one-to-one mapping (also known as correspondence, transformation, or function, all of which are equivalent terms in this context) is crucial to the development of Cantor's theory. In the discussion which ensues, we shall avoid the controversy which has developed around set theory since Cantor's time (it turns out that we have banished the ghost of Zeno in modern mathematics, only at the cost of opening a veritable Pandora's box of new and very mocking ghosts—but more about that later), by adopting what is technically known as the naive approach to set theory, which was that of Cantor himself. Cantor described a set as "a collection into a unity, of definite, distinguishable objects of our intuition or of our thought. These objects are called the elements or members of the set." One must avoid confusing "subset" with "member." Every set is a subset of itself, for example, but no set can be a member of itself. For example, the set whose only element is the number 5, written as {5}, is not the same as the number 5. 5 is a member of {5}, but it is considered nonsense to say {5} is a member of {5}; and yet it is considered true (trivially true, but true nonetheless) to say that {5} is a subset of {5}. This is an example of the formal conventions which we must get used to in discussing set theory, naive or otherwise.

The concept of mappings is very clear with finite sets, whether small or having over a billion elements. It is clear that the idea of a set being more or less numerous than another set, is basically independent of the concept of counting, and of the existence of numbers, in spite of the use of the word "numerous" (we could just as well choose another arbitrary but suggestive term, such as "populous"). We shall now consider familiar finite sets, and make use of the concept of one-to-one mapping, in order to define exactly what we mean by a set being equally numerous, or more numerous , with respect to another set.

Definition: If A is a set and B is a set, then the statement that "A and B are equally numerous" means that there exists a one-to-one mapping of A onto B.

Now, we also want to define the concept of one set being more numerous than another. Suppose we restrict ourselves to finite sets. Then we can make use of the mapping concept thus: the statement that A is "more numerous" than B, means that there exists a one-to-one mapping of B onto some proper subset of A. (A proper subset is simply any subset that is not identical to the set itself, in other words, a non-trivial subset). For example, if A is the set of numbers 1 through 10 inclusive, and B is the set of numbers 31 through 40 inclusive (we really mean counting numbers here), then it is obvious that A and B are equally numerous (or "equivalent" as some prefer to say), since we can easily find a way to map A onto B in the 1-to-1 fashion, so that each element of A is uniquely paired with an element of B, and of course, vice versa. On the other hand, if A is the set of counting numbers 1 through 50 inclusive, and B is the set of numbers 125 through 150 inclusive, then B is less numerous than A since we can map B onto a proper subset of A in a 1-to-1 way—for instance, the proper subset 1 through 25 inclusive. Note that although we are obviously appealing to knowledge of counting and arithmetic through these examples, it is not essential that we do so, as demonstrated by the Hottentot's ability to set up a 1-to-1 mapping between a proper subset of his stones and all his returning cattle, thereby deducing, without using numbers, that some cattle were missing.

Now, this situation is strangely altered when we consider infinite sets. Let us consider the infinite set of all positive integers. Imagine them embedded along the number line, stretched out to infinity:



Now, surely, the set of all even numbers, 2, 4, 6, 8, 10 . . ., etc., is a proper subset of the set of all counting numbers; and yet it is also infinite. Is there a way to map them together 1-to-1? Clearly, there is:

1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**. . . . . . .**
↕	↕	↕	↕	↕	↕	↕	↕	↕	
2	**4**	**6**	**8**	**10**	**12**	**14**	**16**	**18**	**. . . . . . .**

(The general rule for the mapping is: if n is a counting number, then n is mapped onto an even number 2n.)

Here, we have a thorough, perfect 1-to-1 mapping of a set *onto one of its own proper subsets*, something that is impossible for finite sets. There are just as "many" even numbers as counting numbers! We can carry this further: as an exercise, demonstrate to your own satisfaction that the following sets are equally numerous with the set of all counting numbers: a) the set of all counting numbers divisible by 3; b) the set of all counting numbers divisible by 1000; c) the set of all perfect squares, i.e., 1, 4, 9, 16, 25, . . . etc. (incidentally, this mapping was noticed by Galileo, who was rather perturbed by it); d) the set of all integers (positive, negative, or 0); e) the set of all powers of 10.

It turns out that this characteristic of being equally numerous with one of one's own proper subsets is the distinguishing characteristic of all infinite sets. Hence, we still need to find an adequate definition for one set being more numerous than another, regardless of whether the sets are finite or infinite. Cantor developed the following definition, which can be demonstrated to satisfy our intuition of "greater numerousness" for any sets: *If A is a set and B is a set, then the statement that A is more numerous than B, means that there exists a 1-to-1 mapping of B onto some subset of A, but there exists no 1-to-1 mapping of A onto any subset of B*. We deliberately avoid saying "proper subset'' here—it is unnecessary, logically. This is an important definition, bearing careful scrutiny.

However, we still haven't found any set that is demonstrably more numerous than the counting numbers. Could it be that all infinite sets are equally numerous? Cantor set about searching, either for an example, or for an argument, to prove one or the other of these two possibilities: either all infinite sets are equally numerous; or else some infinite sets (such as perhaps the set of all real numbers, or points on a line) are more numerous than others. It turned out not to be easy at all to settle this question, and when, through some of the most original work in the history of mathematics, Cantor did settle it, there were some great surprises. But before we get to these, let us invoke the famous story of the infinite hotel (attributed to the great German mathematician and champion of Cantor, David Hilbert), and hopefully thereby make some of these ideas more vivid.

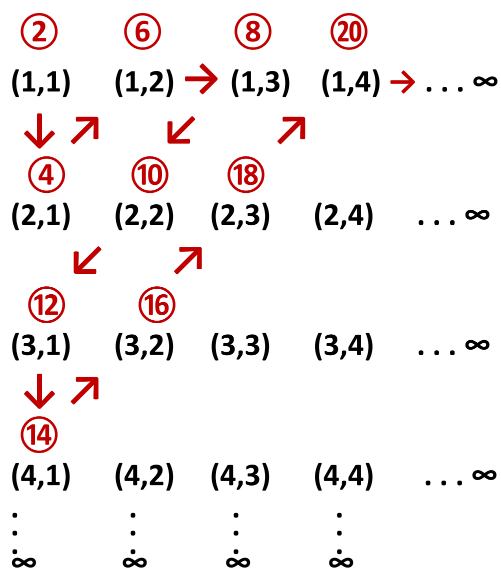
The Hotel Hilbert is a remarkable establishment. It has infinitely many rooms, labeled 1, 2, 3 . . . etc., to infinity. The universe in which it exists (since there might not be enough room for it in the mere physical universe for all we know) is also remarkable, in that its inhabitants can travel or communicate at any distance instantaneously. Thus if the manager of the hotel tells the bellboy to go to Room 3, and take a message from there to deliver to Room 500 million-to-the-millionth power, it is done in a flash, in spite of the fact that the latter room is very far away indeed. Now, on a given day when our story opens, all the rooms of the Hotel Hilbert are full. An important guest arrives, and the desk clerk is upset that there is no room for this guest. But the manager, who is immediately called, is very experienced in such matters. "After all, this is no mere finite hotel," he tells the clerk. He then gives the following order: The guests are all to leave their rooms for just a minute; then the guest who was in Room #1 is to move to Room #2, the guest who was in Room #2 is to take Room #3, and so on. When this is done, all the guests now have new rooms (no one carries luggage in this universe, since whatever one needs can be "materialized" immediately with the excellent accommodations available in each room of the hotel), and Room #1 is vacant. The arriving guest then takes Room #1. The very next day, these rooms are still all occupied, and a whole party of important people arrives. There are 100 million in the party. Absolute privacy is the ironclad rule of this hotel, so there is as usual no question of any guests sharing rooms. But the clerk is intelligent, and has learned the lesson well. Without troubling the manager, he gives the order for all guests to leave for just an instant; he then sends the guest who was in Room #1 to Room #100,000,001, Room #2 to Room #100 million-and-2, and so forth. When they are all settled (a matter of a few seconds), he then has exactly enough rooms for the important party. But the next day another party arrives, even more important than the previous party—and in this party are infinitely many guests. In despair, he calls the manager. "How many times must I tell you, this is no mere finite hotel!" he chides the clerk, "this is a first-class infinite hotel, rated Aleph-null in all the best guides." He then gives orders for the guests to step out for a moment, and places the guest from Room #1 in Room #3; the guest from Room #2 in Room #5, ... and so on.[6] When this is done, there are infinitely many rooms left vacant, namely those with even numbers over the doors. So the situation is resolved again, thanks to the experience of the manager.

All goes well for some time, business is booming, and the hotel is always full. Then one day our clerk receives a telegram. He wishes the manager were there (he has stepped out momentarily), for it is a very important telegram indeed, from the owners of the hotel, who have been experiencing financial difficulties. Now this board of directors owns many other interests besides the Hotel Hilbert; in fact, the Hilbert is only one of a whole chain of such infinite first-class, Aleph-null rated hotels—in fact, an infinite chain of them. The telegram states that, because of the aforementioned financial difficulties, they have demolished the entire chain of hotels, except for the Hilbert. Each of these hotels was full, like the Hilbert. But something must be done with the guests. Accommodations for every single one of these guests must be found at the Hilbert, or the lawsuits that would result would be horrible to contemplate. Furthermore, the guests from the now-demolished infinite chain of infinite hotels will arrive in an hour. His manager is out, and cannot be reached. Our clerk is truly in a panic this time. He consults with every one of the other clerks on duty, some of whom are much more experienced than he, but no one has ever had to deal with a situation like this before. An infinite collection of parties arriving, with infinitely many people in each party! And the hotel is already full. The time of projected arrival is drawing near, the manager still cannot be reached, and the staff is at a loss for a solution. In desperation, someone suggests asking Scharf, a part-time bellhop who is a mathematics student. He is immediately summoned from the Googol sector, and informed of the problem. After thinking for a moment, he comes up with the following plan: "The first thing we must do is assign labels to the guests who are arriving," he says. "Each guest will be assigned two numbers: the first number will be the number of the (now demolished) hotel he or she was staying in, and the second number will be the room number of that hotel which was occupied by the guest. Is this information available?" The clerk assures him it is, and immediately obtains it from the central computer records; Scharf then makes the following table, which gives each of the arriving guests a unique label. The rows of the table each represent the register of one hotel; and the columns represent room numbers. His first number in each ordered pair identifies the hotel the guest was staying in before demolition, and the second number in each ordered pair identifies the room number of that hotel:

[6] In general, the guest from room # n takes room # 2n + 1 as his new room.

		Room # →			
	(1,1)	(1,2)	(1,3)	(1,4)	(1,5) . . . ∞
Hotel #	(2,1)	(2,2)	(2,3)	(2,4)	(2,5) . . . ∞
↓	(3,1)	(3,2)	(3,3)	(3,4)	(3,5) . . . ∞
	(4,1)	(4,2)	(4,3)	(4,4)	(4,5) . . . ∞
	(5,1)	(5,2)	(5,3)	(5,4)	(5,5) . . . ∞
	⋮ ∞	⋮ ∞	⋮ ∞	⋮ ∞	⋮ ∞

"So you see," says Scharf, "each of the arriving guests is now uniquely identified by an ordered pair of counting numbers, and we have it solved." Looking over this table, the clerk says "I see that you have labeled these people—each one is identified by a unique pair of integers, and any given pair of integers must label one and only one guest—very neat. But where the devil are we going to put all these people?" "Oh that," says Scharf. "First, have all the guests now in the Hilbert step out for a moment." Since time is getting short, the order is immediately given and executed. "Now," says Scharf, "have them take over the odd-numbered rooms—you know how it's done." This is immediately ordered, and the infinitely many Hilbert guests are now found to occupy all the odd-numbered rooms. "Now, this leaves us with all the even-numbered rooms vacant, right? And that's an infinite set of rooms, right? So now we just take our little list of all the incoming guests, and assign them each one of the even numbered rooms, as follows."



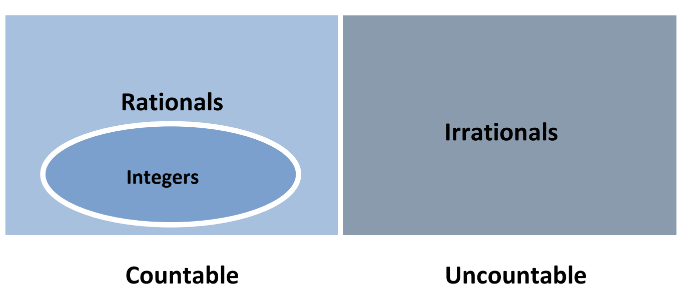
Very skeptically, the staff stares at this room assignment plan. Finally, they have to admit it (after perhaps a good deal of staring): every one of the incoming guests, of whom there are infinitely many parties, and each party containing infinitely many guests, now has a vacant room awaiting them in the Hotel Hilbert. The day is saved, and the manager gives the bellboy a promotion and squares his salary when he hears about this on his return.

Returning to the story of Cantor's researches—one of the first sets Cantor considered as a likely candidate for an infinite set which would be more numerous than the integers, was the set of all rational numbers, meaning the set of all fractions with integers as numerator and denominator. More rigorously, the statement that x is a *rational* number, means that there exists an integer m, and an integer n, such that x = m/n. Now, let us visualize the number line, and see where the rationals are distributed. We leave it as an exercise to the reader to visualize this, and to demonstrate that in fact the rationals are virtually everywhere—no matter where we look along the number axis, no matter how tiny a segment of this axis we pick—for example, all the numbers between zero and 1/100-quadrillion—there will be infinitely many rational numbers in that tiny segment. We might be tempted to conclude that all numbers are rational. In fact, this is what the early Greeks thought at first, until it was demonstrated (by a student of Pythagoras according to one account) that, for example, the diagonal length of a square with side = 1 unit cannot be rational (in modern symbolism, we call this irrational length $\sqrt{2}$)—that there do not exist two integers m and n such that $1^2 + 1^2 = (m/n)^2$.

This discovery was considered shocking at that time. His proof of this fact (which we won't go into at this time) made use of one of the most important tools of mathematical reasoning: the reductio ad absurdum, or "indirect proof," which proves a given proposition by assuming that it is not true, and arriving at a contradiction. In this case, the proof was effected by supposing that there do exist two such integers m and n, and arriving at a conclusion that a positive integer can be both odd and even, an obvious absurdity. Hence the irrational was admitted into mathematics.

In modern decimal terminology, we now have a much more streamlined way of describing both rational and irrational numbers. Every real number, whether rational or irrational, has a unique decimal expansion. It can be proven that a rational number has a decimal pattern which, sooner or later, begins to repeat itself infinitely: for example, 1 = 1.00000000 to infinity; and the number 1/3, obviously also rational, = .3333333 to infinity; more complicated, but equivalent cases, are 1/7 = .142857142857142857. . . etc. to infinity; 8/11 = .72727272 . . . etc. The decimal expansion of an irrational number never repeats itself in this endless fashion—sooner or later, the repetitive pattern must be broken, over and over. To summarize this relation between these different types of number sets, we make use of the familiar Venn diagram:

The Continuum of Real Numbers



Now, to return to Cantor once more, it might be suspected that the rationals are a "larger" set than the integers— after all, they are so thickly populous, so infinitely dense along the number axis, that they are everywhere it seems, whereas, the integers have wide spaces between them. If however, Cantor was able to construct a perfect one-to-one mapping between these rationals (exhausting the set) and the set of all integers. Thus it was proven that there are as "many" integers as there are rational numbers! More properly, the set of all rational numbers is equally numerous with (equivalent to) the set of all integers. Cantor's proof of this fact was virtually identical to the scheme used by our hero Scharf, in assigning rooms to the huge party arriving at the infinite hotel. Each rational number can be labeled with a number pair, as follows: for the number 5/6, we assign the pair (5, 6), and in general, for each rational number m/n, we assign the pair (m, n). After some cosmetic work in getting rid of reducible fractions, such as 6/12 (this is a minor difficulty, since even if we allowed the much larger set of reducible fractions, the argument would be essentially the same), we see that every single one of the rational numbers can be associated with a unique pair of integers, which in turn can be associated with a unique integer. We leave it to the reader to reconstruct Cantor's great proof, using Scharf's plan as the clue.

After finding this one-to-one mapping of all the rational numbers onto the integers, we might imagine that Cantor began to ask himself if, indeed, all infinite sets are equally numerous. His next great discovery was a proof that there can be no 1-to-1 mapping between the integers and the continuum of all real numbers. The argument is one of the classics of modern mathematics, and is now called the "diagonal" method of proof.

Cantor began as follows: suppose that there is a one-to-one mapping, which labels every one of the real numbers, rational and irrational, with a unique integer. If there is such a mapping, then it would look something like this (for simplicity, we suppose that just the numbers between 0 and 1 have been mapped):

Integers		**Real Numbers (in decimal expansion)**
1	***maps to***	**.023439604875193…..**
2	***maps to***	**.928000000395741…..**
3	***maps to***	**.676482088120847…..**
4	***maps to***	**.982415553462768…..**
.	.	.
.	.	.
.	.	.
.	.	.

Assuming that some sort of mapping like this has been found, Cantor then constructed the following decimal number: $a_1, a_2, a_3, a_4 \ldots a_n, a_{n+1} \ldots$ with the following characteristic: each a_k is a digit which differs from the k^{th} digit of the k^{th} number in the above mapping. In other words, he constructed this number by following the diagonal formed by the infinite square array of digits in the above mapping, and making sure that his number differed from that diagonal number at each digit of its expansion. For example, the diagonal digit in the above picture would be .0264 . . . etc., and our constructed digit could be something like .1375 . . . . Now, the digit we have constructed cannot appear anywhere in the infinite list, because it differs from each number in that list; if we take the millionth number in the list, then our number differs from that number in its millionth digit of its decimal expansion; for all we know, it may be exactly the same in every other digit of the two expansions—but it is guaranteed to differ from it in the millionth decimal place and a miss is as good as a mile here!

What this implies is that there can be no one-to-one mapping between the integers and the set of all real numbers between 0 and 1; for any mapping one can concoct, we can always find a number between 0 and 1 which has been left out of the mapping. This suffices to show that there is no such mapping possible, and hence that the set of all real numbers between 0 and 1 is "more numerous" than the set of all integers, and hence more numerous than the set of all rationals. As an exercise, show that there does exist a mapping between a proper subset of the numbers between 0 and 1, and all the integers (hint: try the set of all rational fractions with numerator 1).

Hence we have a curious situation: there are at least two kinds of infinity! The infinity of the integers, which is equivalent to the infinity of the rationals, and the greater infinity of the real numbers (and just those that lie between 0 and 1). It is not difficult at all to show that the infinity (from now on, we shall adopt Cantor's terminology, and use the word cardinality to refer to the "numerousness" of a set) of the real numbers between 0 and 1 is equivalent to that of a much larger segment, or, in fact, between all the real numbers, extended throughout the infinite axis of these numbers. Because of space considerations, we must refer the interested reader to the Bibliography here for an indication of this proof.

How many kinds of infinity are there? To deal with this question, Cantor introduced a new terminology: the Cardinal Number of a set. The cardinal number of a finite set is simply the number of elements in the set. The lowest order of infinite sets is that of the integers; such an infinite set is said to be countably infinite, and its cardinal number is designated with the first letter of the Hebrew alphabet, $\aleph$, or aleph. This cardinality is called $\aleph_0$ or aleph-null. The still higher cardinal number (cardinal numbers of infinite sets are called transfinite numbers) of the continuum of real numbers is called $\mathfrak{c}$.

What about other cardinalities? In the early phase of his researches, Cantor supposed that the points of two-dimensional space would have to be of a higher cardinality than that of the continuum, which is equivalent to the points on a line. Yet the search for a proof that the cardinal number of the points of the Cartesian (or Euclidean) plane is greater than $\mathfrak{c}$, was fruitless. Consulting some of his eminent colleagues, Cantor was actually told that the proof of this fact was "unnecessary, because it is self-evident that there must be a greater cardinality for each higher dimension." Finally, to the great surprise of most mathematicians, Cantor found a perfect 1-to-1 mapping between the one-dimensional continuum of the real numbers, and that of two-dimensional space, three-dimensional space, or any Euclidean space of finite dimensions. Cantor himself was somewhat taken aback at this unexpected result: in a letter to Dedekind, he said "I see it, but I don't believe it."

The search for a transfinite number higher than $\mathfrak{c}$ began to resemble the previous search for a transfinite number higher than $\aleph_0$. Could it be that there are only two infinite cardinal numbers?

At this point, Cantor achieved a powerful generalization, independent of any consideration of specific sets, such as the real number continuum. He proved the following fundamental theorem: given any set whatsoever, whether finite or not, and call this set S. Then the set of all subsets of S (the "power set") is itself a set, and this set has higher cardinality than S. The proof of this theorem is one of the shortest, simplest, and most elegant in all mathematics. However, to present it here would involve some further explanation, and so the interested reader is again referred to the Bibliography. What Cantor's theorem implies is an endless collection of transfinite numbers, each greater than the others which precede it.

To deal with this bewildering infinity of infinities, Cantor introduced the further concept of ordinal number, differing from cardinal number in that it carries along with it a notion of preceding, succeeding, and in short, order. The lowest infinite ordinal number, that of the integers ordered 1,2,3,4 . . ., is called by the Greek letter ω or omega. We shall summarize the results of these researches (again, we refer readers to the Bibliography) by noting that Cantor (by now aided by other eminent mathematicians, such as Zermelo), was able to prove that all of the "alephs" can be subjected to a process known as well-ordering, a type of ordering in which we are guaranteed that each term has an immediate successor. Hence, though it still remains a somewhat mysterious question as to "how many" transfinite numbers there are (most contemporary mathematicians would probably challenge the meaningfulness of this question), we can at least guarantee that: a) there is a smallest transfinite, namely $\aleph_0$ and b) $\aleph_0$ has an immediate successor, $\aleph_1$, the next greatest transfinite, and we can continue this series $\aleph_0$,$\aleph_1$, $\aleph_2$, . . . indefinitely.

But what of the cardinal number $\mathfrak{c}$? The attentive reader may well have wondered why the cardinality of the real number continuum was not called $\aleph_1$ instead of $\mathfrak{c}$. In other words, does not $\aleph_1 = \mathfrak{c}$? The answer to this could not be found. Nor is it found today. This is the famous "continuum hypothesis," which can be given the equivalent formulation: Does there exist a transfinite number which is greater than $\aleph_1$ but less than $\mathfrak{c}$? It seemed to Cantor that no such transfinite number should exist, and most mathematicians concurred in this intuition. But the Pandora's Box of new ideas which his researches had opened began to challenge the very nature of our mathematical intuition.

The first paradoxes of set theory appeared right on the heels of Cantor's triumphs. The most famous is probably Bertrand Russell's antinomy, which goes as follows: Consider the set of all sets. This set, if we allow it to exist (and why shouldn't we?), violates the principle that no set can be a member of itself, since it is a set. Let us call this an "extraordinary" set, and let us call an "ordinary" set one which is not a member of itself, such as the set of all real numbers, etc. Now, if we wish to preserve the principle that no set can be a member of itself, then we shall surely have to consign all such "extraordinary" sets to the limbo of nonsensical entities. However, consider the set of all ordinary sets. Is this set an ordinary set? Is it not a member of itself? A little careful deductive reflection leads to the inescapable paradox that if this set is a member of itself, then it is not ordinary, and hence cannot be a member of the set of all ordinary sets—that is, it cannot be a member of itself; and if it is not a member of itself, then it is an ordinary set, and hence must be a member of itself, since it is the set of all ordinary sets.

Such a contradiction cannot be admitted into mathematics. Mathematicians at once got rid of this contradiction by saying: it is meaningless to speak of such things as the set of all sets, or the set of all ordinary sets, or in fact, the set of all cardinal numbers. Hence we cannot assume that all definitions of sets are meaningful. But this left an uneasy feeling.

Since those days many new developments have taken place, both in set theory and in logic, and in the logical analysis of axiomatic systems, which is known as meta-mathematics. A bewildering proliferation of logical systems, notations, philosophies, and strange new entities crowd the modern scene, which seems to resemble something from *Alice in Wonderland*, if not the Tower of Babel. The most important and significant of these post-Cantorian results is from the meta-mathematical work of Kurt Gödel and P. J. Cohen. Gödel was able to prove, assuming only the validity (i.e., freedom from self-contradiction) of the axioms of arithmetic, that no mathematical system can be complete; in other words, given any system of axioms strong enough to capture arithmetic, there will always exist a statement, defined within that system, which is independent of the system, that is to say, which can never be proven or disproven by the other axioms in the system. The famous Fifth, or “parallel” postulate of Euclid was known to be an example of such an independent axiom—by assuming it is true, we get the rich and consistent system of Euclidean space; by assuming it is false, we get either of two other types of space, equally rich and consistent. At this point, Gödel and other mathematicians began to wonder whether the continuum hypothesis, which states that there is no cardinal number between $\aleph_0$ and $\mathfrak{c}$ might be such an unprovable or independent proposition. Finally, in 1963, Cohen, working from the base provided by Gödel, apparently proved that the continuum hypothesis is in fact unprovable, and independent of the other number-system axioms. This means that we can have two types of number systems: one in which there are no sets more numerous than the integers but less numerous than all real numbers, and a different system in which such intermediate cardinals do exist. At this point, we are very far indeed from the classic simplicity of the Greeks, and the notion of mathematical truth seems to have become questionable. This situation has given rise to schools of mathematics which have very differing philosophies about what constitutes a valid proof and what does not. Even that time-honored principle of Aristotelian logic, the Law of the Excluded Middle (which says that if X is any logical-mathematical proposition, then either X is true, or else it is not true), has now come under fire by certain schools of mathematics.

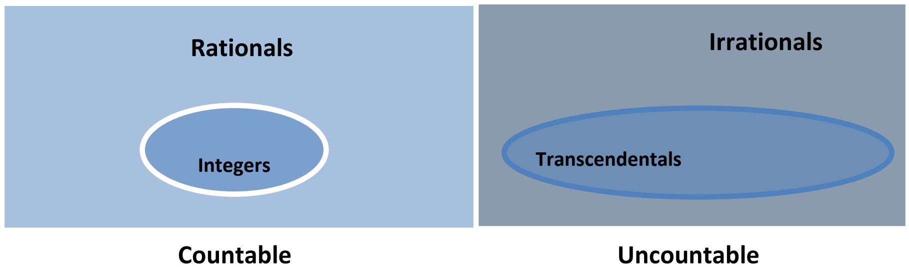
But we do not need to examine all these schools of mathematical philosophy (who sometimes disagree with each other in a very emotion-charged way), in order to make use of certain of these notions as symbols of the relationship between ordinary and higher consciousness. To one such as Merrell-Wolff (and presumably the reader), who has become attuned to the numinous presence of a Consciousness which goes beyond the greatest powers of the discursive intellect, such ideas as the transfinite alephs are valuable, not as subjects of hot debate as to their “truth,” but as very sublime symbols. To the mystic, no symbol or system of thought can ever be adequate to describe the greater Reality in any case, and such debates appear as empty. But we must use words to communicate (at this stage of our evolution at any rate) with each other. To communicate some sense of the Higher Consciousness, poets, philosophers, and mystics have hit upon many devices, from the koans of Zen, to the ravishing and extravagant hyperbole of Sufi poetry, to the sutras, poetry, and philosophy of the great religious traditions. Of course none of these devices will ever capture It—but the search for new and suggestive formulations goes on, and is well worth undertaking. The work of Merrell-Wolff has been particularly remarkable in making use of the notions of the transfinite as very striking symbols of the higher consciousness.

We conclude this discussion by examining another result of Cantor's which Merrell-Wolff refers to in his lectures, and which we have deliberately not discussed until now. It is the notion of transcendental numbers. This is a technical term which describes a certain class which forms part of the numbers called "irrational": a transcendental number is one which is not found as the root of an equation of the form: $a_0x^n + \ a_1x^{n-1} + a_2x^{n-2} + \ldots a_n = 0$, where n is a counting number and each $a_0, a_1, a_2, \ldots\ a_n$ is a fixed integer, positive, negative or zero.

Now, it will demand quite a bit of algebraical expertise to understand the way this equation works. Most readers may not have this facility, and so once again, because of limited space, we refer them to the Bibliography, especially to the excellent book, *Stories about Sets*, by N. Y. Vilenkin. However, it is not necessary to fully understand the previous general equation in order to appreciate what follows. Now, any number x, which satisfies an equation of this sort, is called an algebraic number. Virtually all the irrational numbers we know of, such as $\sqrt{2}$, $\sqrt[3]{93}$, turn out to be algebraic. Furthermore, all the integers and rational numbers are algebraic. It was thought at first that all numbers of the real continuum are algebraic. But, long before Cantor began his researches, it was discovered that some numbers cannot be algebraic. These numbers were designated as transcendental. Much later, it was proven that π is such a transcendental number. Furthermore, any number taken to an irrational power is transcendental. We do not know why the mathematician who originally designated such numbers chose the word 'transcendental' and we may well wonder if something deeper than merely choosing a new name was not working in his consciousness, in light of what Cantor discovered about these numbers.

We begin by recalling that Cantor had discovered that the integers, together with the rational numbers, are countably infinite (i.e., of cardinality $\aleph_0$), and that he had proved, with his diagonal method, that the set of all real nuribers is uncountably infinite (i.e., of cardinality higher than $\aleph_0$. The question arose: How much of the continuum can we remove until there remains only a countably infinite set? Or, phrased another way, how many sets of numbers can we keep adding to the integers until we get a set that is uncountably infinite? To begin with, the set of all irrational numbers is uncountably infinite; in fact, if any countably infinite set of numbers is removed from an uncountably infinite set, the remaining numbers remain uncountably infinite. How do the transcendental numbers fit into this? To help visualize this, we revise our previous Venn diagram of the continuum to now include the "transcendental" designation as follows:

The Continuum of Real Numbers



The natural question arises: Are the transcendental numbers countably infinite, or uncountably infinite? And the further question, are the algebraic irrational numbers (that is, the non-transcendental irrationals, which are the blue-shaded irrationals in the diagram) countably infinite? Cantor was able to prove the following: all non-transcendental numbers are countably infinite! Or more properly phrased, the set of all algebraic (= non-transcendental) numbers is a countable set. This means, rather loosely phrased, that the infinity of the transcendental numbers, represented by the deceptively small shaded kernel in the diagram, is the essence of the “stronger” uncountable infinity of the entire continuum. If we remove this transcendental kernel, we are left with only the “weak” infinity of the algebraic numbers, and we could never rearrange these numbers to retrieve the higher, uncountable infinity of the continuum. On the other hand, if we were to remove all the rational and irrational non-transcendental numbers, the remaining transcendental kernel would be quite sufficient in “strength” to enable us to reconstruct a new continuum, by re-arranging these numbers. In a way, this was far more surprising than the previous result that the rationals are countable.

We have covered quite a bit of conceptual material here. It will be noticed that, as one follows these ideas, a curious, exhilarating, and perhaps even scary feeling arises into consciousness: the feeling of one’s imagination being stretched to the breaking point. Now, it is just this feeling which suggests the value of these ideas as symbols. In previous times, the poet-mystic Rumi used a peculiar type of ecstatic poetry to stretch the imagination of his readers to the breaking point, and thus aid a certain adumbration, if not the Awakening, of a consciousness which transcends the utmost powers of subject-object thought, analogous to the way the inaccessible endlessness of the higher alephs transcends our initial concept of infinity. It is hoped that the reader will now have some idea of how valuable is Merrell-Wolff’s contribution, in suggesting that the symbol of the transfinite, as well as other ideas from pure mathematics, is peculiarly appropriate for our modern-day phase of consciousness, in the same way that Rumi’s magnificent poetry was appropriate for his age.

IV. A Note on Pre-Greek Mathematics

As far as we know, the notion of demonstrative proof by pure logical deduction was virtually absent in the systems of mathematics which preceded the Greeks, such as those of the Egyptian-Chaldean civilization. However, as Rudolf Steiner said, modern man attains his greatest stupidity when he sees ancient man as stupid. A rather startling example of great mathematical sophistication among ancient men is afforded by a study of the dimensions of the Great Pyramid. Assuming no development of the axiomatic method among the designers of this pyramid (and there is no evidence of such development), we can only marvel at the mathematical and scientific knowledge encoded in this structure. Here is but one example (discussed in several of Merrell-Wolff's audio-taped lectures) of this encoding.[7]

Suppose we are standing before the Great Pyramid in Egypt. There is no convenient way to measure the height, but there are many easy ways of obtaining the angle of slope of one of the faces. Now, when this is actually measured, it turns out to be 51˚51'14.3". What does this figure signify? Let us jump ahead of the story, and suppose that a guide happens along at this point, and makes the following assertion: "Did you know that the height of this pyramid is equal to the radius of a circle, whose circumference is equal to the perimeter of the square base of this pyramid?" Now, if we remember enough of our high-school geometry, and a little elementary trigonometry, we can test this assertion, knowing the angle which we have just measured. And if we know something about the history of mathematics, we might well be inclined to disbelieve the guide entirely, since the ability to know with any accuracy the radius of a circle with circumference equal to a rational perimeter of given length, would imply an ability to calculate the value of the transcendental number π very accurately, something ancient peoples simply had no way of doing as far as we know from extant records.

The sun is very hot, and we don't feel like pacing off the length of the base. It is an enormous pyramid, after all. In fact, the guide has also pointed out to us that some of the present villages in the vicinity of the pyramid have been built entirely from stones pilfered from the pyramid! The sheer size of this monument gives rise to another reflection: the value of π would have to be of an accuracy comparable to that which modern scientists need in everyday work, if great errors were not to show up in such enormous dimensions. A little more geometrical reflection shows us that there is no need to measure any other dimensions than the angle of 51˚51'14.3" in order to test the guide's assertion. Given a pyramid of square base, and this angle of facial slope, a unique shape of pyramid is determined.

In the sketch of the pyramid below, the triangle outlined heavily turns out to be the essential figure we want to analyze. It is a right triangle whose height (**h**) is the height of the pyramid; the base (**b**) of this triangle is one half the length of one side of the square base of the pyramid, which in turn is the same as one eighth of the perimeter; the hypotenuse of this triangle is the length of a face of the pyramid measured from the cap to the center of a base;

[7] See, for example, Franklin Merrell-Wolff, "Mathematics, Philosophy, & Yoga," Part 4 (Phoenix, AZ: November 20, 1966).

and finally, the angle between **a** and **b** is our angle 51˚51′14.3″, which we call **ϴ** for convenience.

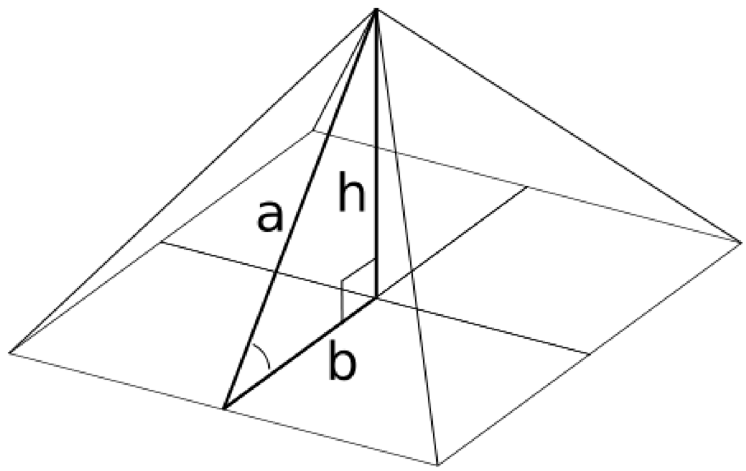
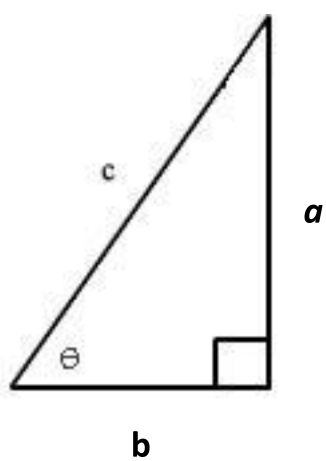


Diagram of the Great Pyramid



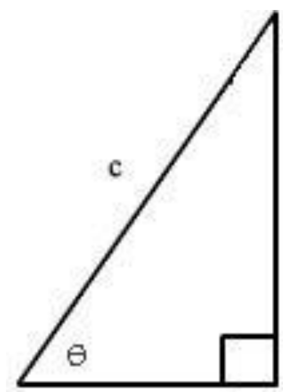
***ϴ** = 51˚51′14.3″*

Now, if P denotes the perimeter of the base, then clearly:

$$b = 1/8\, P.$$

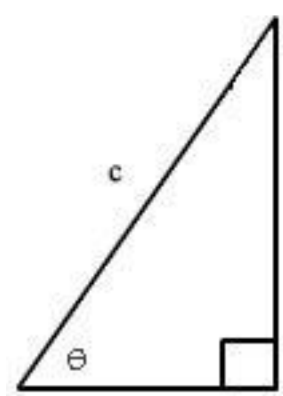
The guide has claimed something about **a**; namely, that **a** is the radius of a circle with circumference P. From geometry, we know this is equivalent to saying:

Claim: $\boldsymbol{P = 2\pi a}$. Back to our triangle:



b = ⅛ P

Let us set b = 1 unit of any measurement whatever, and see what the consequences are.



b = 1

Since $\boldsymbol{b = 1 =}$ ⅛$\boldsymbol{P}$, from equation (1), then ⅛$\boldsymbol{P = 1}$, and hence $\boldsymbol{P = 8\,units}$.

Now, the guide claims that $\boldsymbol{P = \pi A}$, hence that $\boldsymbol{8 = 2\pi a}$ *;* now using elementary algebra, we divide both sides of this equation by 2**a**, and obtain the final interpretation of the guide's claim:

Claim: $\boldsymbol{\pi = 4/a}$.

Now, in fact, we can actually calculate A in unit terms, from trigonometry, using our measured angle of 51˚51′14.3″—the ratio of A to B (= 1) is called the tangent of the angle ϴ; and making use of the important tool of modern trigonometric tables, we can actually look up this tangent; so we have $\boldsymbol{tan\ \theta = a/1 = A;}$ **a**nd from the table (or a good hand calculator), we find that $\tan(51˚51''14.3'') = 1.2732395 = \mathbf{a}$. So now the claim reduces to: $\boldsymbol{\pi} = \mathbf{a}/1.2732395$; using a hand calculator, this comes to 3.1415928; the same hand calculator tells me that $\boldsymbol{\pi} = 3.1415927\ldots$. This means that the designers of this pyramid knew the value of π out to the sixth decimal place, in order to construct a facial angle of that accuracy. This provides a very startling confirmation of the guide's assertion, unless one was to take the suggestion of "coincidence" seriously. Now, compared to the age of this pyramid, it is only in relatively recent times that we have been able to calculate π with this sort of accuracy, and our methods are the direct outgrowth of the Greek insight which we discussed previously. Who designed this pyramid? When was it built? And how were they able to make such calculations, seemingly impossible to achieve by mere empirical measurement? These are questions open to much imaginative speculation, but there would appear to be no convincing answers.

Addendum

After writing this, I conversed with a friend who has studied the dimensions of the Great Pyramid in far more detail than I. In looking this over, he claimed that the accuracy of the angle mentioned here is not quite the case, and that the real story of this Pyramid is far more complex than this. However, he was not able to offer any specific corrections, but only this opinion, vaguely drawn from his readings. Certainly this is a subject on which different writers have very different opinions. This is a good place to note to the reader of this Supplement, that I would welcome any informed additions to any of the material in this or other sections. In fact, I should be most happy if this Supplement were to become a collective work of the Sangha, any able students being more than welcome to add more sections to this opus. It is partly for this reason that I have divided this work into sections I, II, III, and IV, and numbered pages separately in each section. May this Supplement grow further, if anyone has something to offer.[8]

And as for the problem of dissonant opinion, whether relating to the Great Pyramid, or to other matters, I would like to follow the lead of the mathematician R. F. Jolly, and conclude this work with the following quotation attributed to Gautama Buddha, and said to be his last instruction to his disciples: "Believe nothing merely because you have been told it. Do not believe what your teacher tells you merely out of respect for the teacher, his age or wisdom. But if after due examination and analysis, you find it to be kind, conducive to the benefit and welfare of all beings, then take that doctrine as your guide."

[8] Please send additions and comments to contact@merrell-wolff.org.

V. Bibliography

Section I

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Section II

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Section III

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